PARTIAL LOCALIZATION, LIPID BILAYERS, AND THE ELASTICA FUNCTIONAL

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Abstract. *Partial localization* is the phenomenon of self-aggregation of mass into high-density structures that are thin in one direction and extended in the others. We give a detailed study of an energy functional that arises in a simplified model for lipid bilayer membranes. We demonstrate that this functional, defined on a class of two-dimensional spatial mass densities, exhibits partial localization and displays the 'solid-like' behavior of cell membranes.

Specifically, we show that density fields of moderate energy are partially localized, *i.e.* resemble thin structures. Deviation from a specific uniform thickness, creation of 'ends', and the bending of such structures all carry an energy penalty, of different orders in terms of the thickness of the structure.

These findings are made precise in a Gamma-convergence result. We prove that a rescaled version of the energy functional converges in the zero-thickness limit to a functional that is defined on a class of planar curves. Finiteness of the limit enforces both optimal thickness and non-fracture; if these conditions are met, then the limit value is given by the classical Elastica (bending) energy of the curve.

1. Introduction

In this paper we study the asymptotic expansion as $\varepsilon \to 0$ of the functional

$$\mathcal{F}_\varepsilon(u,v) := \begin{cases} \varepsilon \displaystyle\int |\nabla u| + \frac{1}{\varepsilon} d_1(u,v) & \text{if } (u,v) \in \mathcal{K}_\varepsilon, \\ \infty & \text{otherwise.} \end{cases} \tag{1.1}$$

Here $d_1(\cdot,\cdot)$ is the Monge-Kantorovich distance (Definition 3.1), and

$$\mathcal{K}_\varepsilon := \Big\{ (u,v) \in \mathrm{BV}\big(\mathbb{R}^2; \{0,\varepsilon^{-1}\}\big) \times L^1\big(\mathbb{R}^2; \{0,\varepsilon^{-1}\}\big) : \int u = \int v = M,\ uv = 0 \text{ a.e.} \Big\}. \tag{1.2}$$

How this singular-perturbation problem is related to the terms in the title—partial localization, lipid bilayers, and the Elastica functional—we explain in the rest of this introductory section.

1.1. **Lipid Bilayers.** Lipid bilayers, biological membranes, are the living cell's main separating structure. They shield the interior of the cell from the outside, and their mechanical properties determine a large part of the interior organization of living cells. The main component is a lipid molecule (Fig. 1) which consists of a head and two tails. The head is usually charged, and therefore hydrophilic,

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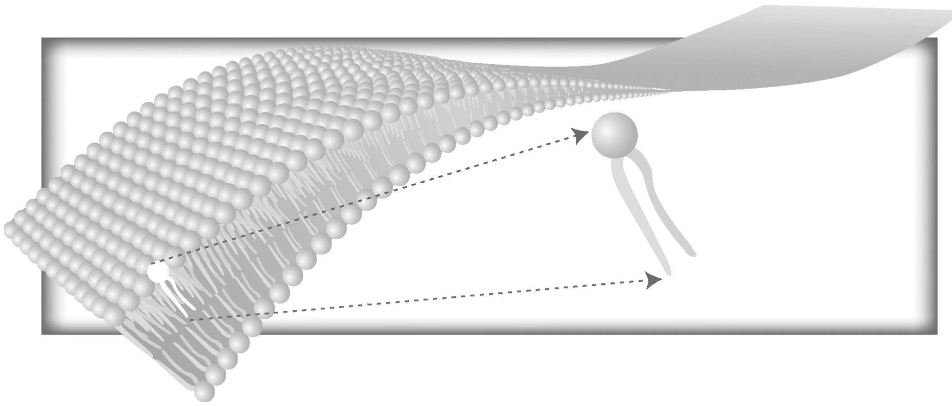

FIGURE 1. Lipid molecules aggregate into macroscopically surface-like structures

while the tails are hydrophobic. This difference in water affinity causes lipids to aggregate, and in the biological setting the lipids are typically found in a bilayer structure as shown in Fig. 1. In such a structure the energetically unfavorable tail-water interactions are avoided by grouping the tails together in a water-free zone, shielded from the water by the heads. Note that, despite the structured appearance of the bilayer, there is no covalent bonding between any two lipids; the bilayer structure is entirely the result of the hydrophobic effect.

As planar structures, lipid bilayers have a remarkable combination of solid-like and liquid-like properties. On one hand they resist various types of deformation, such as extension, bending, and fracture, much in the way a sheet of rubber or another elastic material does: for instance, in order to stretch the bilayer, a tensile force has to be applied, and when this force is removed the structure will again relax to its original length. With respect to in-plane rearrangements of the lipid molecules, however, the material behaviour is viscous, and there is no penalty to large in-plane deformations.

While this interesting combination of properties is clearly related to the chemical makeup of the lipids—most importantly, the hydrophobic character of the tails—quantitative and detailed understanding of the phenomenon is still lacking. Here we focus on a simple question that has already been alluded to above: how can we understand the stability of these planar structures, and their pseudo-solid behaviour, if they are constructed from independent, non-bound molecules? This is the main question behind the analysis of this paper.

1.2. **Partial localization.** The self-aggregation of lipids into bilayers is fundamentally different from the self-aggregation of 'simple' hydrophobic compounds in water. Lipid bilayers are *thin structures*, in the sense that there is a separation of length scales: the thickness of a lipid bilayer is fixed to approximately two lipid lengths, while the in-plane spatial extent is only limited by the surroundings. By contrast, 'simple' hydrophobic compounds aggregate in water to form drops that lack the small intrinsic length scale of lipid bilayers.

We use the term *partial localization* for the self-aggregation into structures that are thin in one or more directions and 'large' in others. The word *localization* is

taken from the literature of reaction-diffusion equations, in which localized solutions are those that are concentrated, in a well-defined way, in a small neighbourhood of a point. By *partial* localization we refer to localization to the neighbourhood of a set that has intermediate dimension, *i.e.* dimension larger than zero (a point) but smaller than that of the ambient space.

1.3. **Energy on a mesoscale: The functional $\mathcal{F}_\varepsilon$.** The functional $\mathcal{F}_\varepsilon$ that is defined above is the result of an attempt to capture enough of the essence of lipid bilayers to address the issues of stability and pseudo-solid behaviour while keeping the description as simple as possible. The derivation of $\mathcal{F}_\varepsilon$, which is given in Appendix A, starts with a simple two-bead model of lipid molecules, inspired by well-known models of block copolymers [35, 10, 29], and proceeds to reduce complexity through a number of sometimes radical simplifications. Despite these simplifications the physical origin of the various elements of $\mathcal{F}_\varepsilon$ remains identifiable:

- The functions u and v represent densities of the (hydrophobic) tail and (hydrophilic) headb eads;
- The term $\int |\nabla u|$, coupled with the restriction to functions u and v that take only two values, and have disjoint support, represents an interfacial energy that arises from the hydrophobic effect;
- The Monge-Kantorovich distance $d_1(u, v)$ between u and v is a weak remnant of the covalent bonding between head and tail particles.

The parameter ε appears in (1.2) as a density scaling. We prove that ε in fact is related to the *thickness* of structures with moderate energy.

We show in this paper that the functional $\mathcal{F}_\varepsilon$ favours partial localization, *i.e.* that pairs (u, v) for which $\mathcal{F}_\varepsilon(u, v)$ is not too large are necessarily partially localized. We will also show that $\mathcal{F}_\varepsilon$ displays three additional properties that were already mentioned: resistance to stretching, to fracture, and to bending. To describe the last of these in more detail we briefly return to biophysics.

1.4. **Energy on the macroscale: The Helfrich Hamiltonian and the Elastica functional.** Intrigued by the shape of red blood cells Canham, Helfrich, and Evans pioneered the modelling of lipid bilayer vesicles by energy methods [17, 31, 24]. The name of Helfrich is now associated with a surface energy for closed vesicles, represented by a smooth boundaryless surface S, of the form

$$E_{\mathrm{Helfrich}}(S) = \int_S \big[k(H - H_0)^2 + \overline{k}K\big]\, d\mathcal{H}^2. \tag{1.3}$$

Here k, $\overline{k}$, and H_0 are constants, H and K are the (scalar) total and Gaussian curvature, and $\mathcal{H}^2$ is the two-dimensional Hausdorff measure. This energy functional, and many generalizations in the same vein, have been remarkably successful in describing the wide variety of vesicle shapes that are observed in laboratory experiments [45]. If we add an assumption of symmetry, in which both 'sides' of the surface have the same properties, the 'spontaneous curvature' H_0 vanishes. For closed surfaces in the same homotopy class, by the Gauß-Bonnet Theorem, E_{Helfrich} is essentially equal to the *Willmore functional* [51]

$$W(S) = \frac{1}{2}\int_S H^2\, d\mathcal{H}^2, \tag{1.4}$$

which arises in a wide variety of situations other than that mentioned here.

In this paper we discuss the above mentioned bilayer models in two space dimensions, *i.e.* we consider functions whose support resembles fattened curves in $\mathbb{R}^2$; in the limit $\varepsilon \to 0$ the thickness of these curves tends to zero. If we consider the limit curves to be two-dimensional restrictions of cylindrical surfaces, we observe that the Gaussian curvature K vanishes. If we also assume that the bilayer is symmetric then the Helfrich energy reduces to the Elastica functional $\mathcal{W}$, the classical bending energy of the curve,

$$\mathcal{W}(\gamma) = \frac{1}{2}\int_\gamma \kappa^2 \, d\mathcal{H}^1, \tag{1.5}$$

where κ is the curvature of the curve. This functional has a long history going back at least to Jakob Bernoulli; critical points of this energy are known as *Euler elastica*, see [49] for a historical review.

1.5. **The main result: the singular limit $\varepsilon \to 0$.** In the main result of this paper we study the limit $\varepsilon \to 0$, and connect all the aspects that were discussed above. The result revolves around a rescaled functional

$$\mathcal{G}_\varepsilon = \frac{\mathcal{F}_\varepsilon - 2M}{\varepsilon^2},$$

where M is the mass of u and v (see (1.2)).

Take any sequence $(u_\varepsilon, v_\varepsilon)_{\varepsilon>0} \subset \mathcal{K}_\varepsilon$. *If the rescaled energy $\mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon)$ is bounded*, then

(1) the sequence u_ε converges as measures;
(2) the limit measure can be represented as a collection of curves (which may overlap);
(3) the support of u_ε is of 'thickness' approximately 2ε;
(4) in particular the *total length* of the curves is $M/2$;
(5) each curve in the collection is *closed*.

Boundedness of $\mathcal{G}_\varepsilon$ along the sequence therefore implies partial localization: the support of u_ε resembles a tubular ε-neighbourhood of a curve. In addition, *stretching* and *fracture* are also represented in this result: stretching corresponds to deviation in total length from the optimal value $M/2$, and fracture to creation of non-closed curves. Neither is possible for a sequence of bounded $\mathcal{G}_\varepsilon$; both are penalized in $\mathcal{G}_\varepsilon$ at an order larger than $O(1)$, or equivalently, at an order larger than $O(\varepsilon^2)$ in $\mathcal{F}_\varepsilon$.

The resistance to *bending* arises as the Gamma-limit of the functional $\mathcal{G}_\varepsilon$ itself:

(1) For any sequence $(u_\varepsilon, v_\varepsilon)_{\varepsilon>0} \subset \mathcal{K}_\varepsilon$,
$$\mathcal{W}(\Gamma) := \sum_{\gamma\in\Gamma} \mathcal{W}(\gamma) \le \liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon),$$
where Γ is the collection of limit curves introduced above, and $\mathcal{W}(\gamma)$ is the curve bending energy defined in (1.5);
(2) For any given collection of curves Γ, there exists a sequence $(u_\varepsilon, v_\varepsilon)_{\varepsilon>0} \subset \mathcal{K}_\varepsilon$ such that $\mathcal{W}(\Gamma) = \lim_{\varepsilon\to 0} \mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon)$.

The details of this result are given as Theorem 4.1.

Summarizing, only partially localized structures can have moderate energy, and such thin structures also display resistance to bending, stretching, and fracture.

So far we have described the result that we prove in this paper. The proof, however, also suggests a more refined result, that we present here as a conjecture:

if a sequence $(u_\varepsilon, v_\varepsilon)_{\varepsilon>0} \subset \mathcal{K}_\varepsilon$ converges to a collection of curves Γ, then

$$\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \;\geq\; 2M + \mathfrak{L}_M(\Gamma) + \varepsilon\mathfrak{B}(\Gamma) + \varepsilon^2 \sum_{\gamma\in\Gamma} \mathcal{W}(\gamma) + O(\varepsilon^3). \tag{1.6}$$

In this development,

- the function $\mathfrak{L}_M(\Gamma)$ is a measure of the deviation of $\sum_{\gamma\in\Gamma} \text{length}(\gamma)$ from length $M/2$: $\mathfrak{L}_M(\Gamma) \geq 0$, and $\mathfrak{L}_M(\Gamma) = 0$ only if this sum equals $M/2$;
- $\mathfrak{B}(\Gamma)$ is zero if all curves $\gamma \in \Gamma$ are closed, and strictly positive (a positive constant times the number of endpoints) if at least one $\gamma \in \Gamma$ is open.

As for the Gamma-convergence above, the inequality in this statement should be interpreted as a 'best-possible' behaviour; while for given limit curves γ sequences exist for which this inequality is an equality, it is also possible to construct sequences along which the difference between the left- and right-hand sides is unbounded.

1.6. **Strategy of the analysis and overview of the paper.** At first glance it is not obvious that the functionals $\mathcal{F}_\varepsilon$ encode bending stiffness effects. Curvature terms in fact appear at order ε^2 of an asymptotic development, which asks for a careful analysis. Most information is hidden in the nonlocal Monge-Kantorovich distance term. The special property that makes our analysis work is that this distance decouples into one-dimensional problems. This reduction is the major step in our analysis and the connection to the *optimal mass transport problem* is the most important tool here. Whereas the idea becomes very clear in simple situations which resemble the lim-sup construction in the Gamma-convergence proof, some effort is required to prove the lim-inf estimate in the general case, where certain 'defects' such as high curvature and vanishing thickness of approximating structures have to be controlled simultaneously.

In the next section we motivate our choice of scaling in the functional $\mathcal{F}_\varepsilon$ and illustrate the behaviour of $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\varepsilon$ in two particular examples. In Section 3 we give definitions of the Monge-Kantorovich distance and 'systems of curves' which describe the class of limit structures. A precise statement of our results is given in Section 4. We briefly review the optimal mass transport problem in Section 5. In Section 6 we provide a short guide to the structure of the proof of our main result, Theorem 4.1, and the proof itself is given in Sections 7 and 8. We conclude in Section 9 with some discussions of generalisations and the wider implications of this result.

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Summary of notation

$\mathcal{F}_\varepsilon(\cdot,\cdot)$	functionals describing the mesoscale energy	(1.1)
$\mathcal{G}_\varepsilon(\cdot,\cdot)$	rescaled functionals	(4.1)
$\mathcal{K}_\varepsilon$	domain of $\mathcal{F}_\varepsilon$, $\mathcal{G}_\varepsilon$	(1.2)
$\mathcal{W}(\gamma)$	bending energy of a curve γ	(1.5)
$\mathcal{W}(\Gamma)$	generalized bending energy of a system of curves Γ	(3.9)
$d_1(\cdot,\cdot)$	Monge-Kantorovich distance	(3.2)
$d_p(\cdot,\cdot)$	p-Wasserstein distance	(3.5)
$\operatorname{supp}(\Gamma)$	support of a system of curves Γ	Def. 3.3
$\theta(\Gamma,\cdot)$	multiplicity of a system of curves Γ	Def. 3.3
$\lvert\Gamma\rvert$	total mass of a system of curves Γ	Def. 3.3
$\mathrm{Lip}_1(\mathbb{R}^2)$	Lipschitz continuous functions with Lipschitz constant 1	(5.3)
$\mathcal{T},\mathcal{E}$	transport set and set of endpoints of rays	Def. 5.4
E	$\{s:\gamma(s)$ lies inside a transport ray$\}$	Def. 7.3
$\theta(s)$	ray direction in $\gamma(s)$	Def. 7.3
$L^+(s),L^-(s),l^+(s)$	positive, negative and effective ray length in $\gamma(s)$	Def. 7.3
$\psi(\cdot,\cdot)$	parametrization	Def. 7.3
E_δ	boundary points with uniformly bounded ray lengths	(7.37)
$\alpha(s),\beta(s)$	direction of ray and difference to tangent at $\gamma(s)$	(7.44)
$\mathfrak{m}(s,\cdot)$	mass coordinates	(7.52)
$\mathfrak{t}(s,\cdot)$	length coordinates	(7.54)
$M(s)$	mass over $\gamma(s)$	(7.53)
E_i,θ_i,ψ_i, L_i^+,L_i^-,l_i^+, $\alpha_i,\beta_i,\mathfrak{m}_i,\mathfrak{t}_i,M_i$	corresponding quantities for a collection $\{\gamma_i\}$	Rem. 7.4
$E_{\varepsilon,i},\theta_{\varepsilon,i},\psi_{\varepsilon,i}$, $L_{\varepsilon,i}^+,L_{\varepsilon,i}^-,l_{\varepsilon,i}^+$ $\alpha_{\varepsilon,i},\beta_{\varepsilon,i},\mathfrak{m}_{\varepsilon,i},\mathfrak{t}_{\varepsilon,i},M_{\varepsilon,i}$	corresponding quantities for a collection $\{\gamma_{\varepsilon,i}\}$	Rem. 7.4

2. Heuristics: the functional $\mathcal{F}_\varepsilon$

In the preceding section we postulated that the energy functional $\mathcal{F}_\varepsilon$, defined in (1.1), favours partial localization. More precisely, we claim that for small but fixed $\varepsilon>0$ functions (u,v) with moderate $\mathcal{G}_\varepsilon(u,v)$ will be partially localized, in the sense that their support resembles one or more fattened curves.

In this section we first provide some heuristic arguments to support this claim, mostly to develop some intuitive understanding of the functional $\mathcal{F}_\varepsilon$ and its properties.

For the discussion below it is useful to remark that the distance $d_1(u,v)$ as defined in Definition 3.1 scales as the mass $\int u=\int v$ times a length scale, and can be interpreted as a u-weighted spatial distance between the mass distributions u and v.

2.1. Fixed thickness ($\varepsilon = 1$), part 1: comparing disc and strip structures.

To gain some insight into the properties of $\mathcal{F}_\varepsilon$ we start with the case of fixed $\varepsilon=1$, and study the limit of large mass

$$M=\int u=\int v,\qquad M\longrightarrow\infty.$$

We now compare two different specific realizations, a disc and a strip.

(1) The **disc:** we concentrate all the mass of u into a disc in $\mathbb{R}^2$, of radius $R\sim M^{1/2}$, surrounded by the mass of v in an annulus (Figure 2). In this

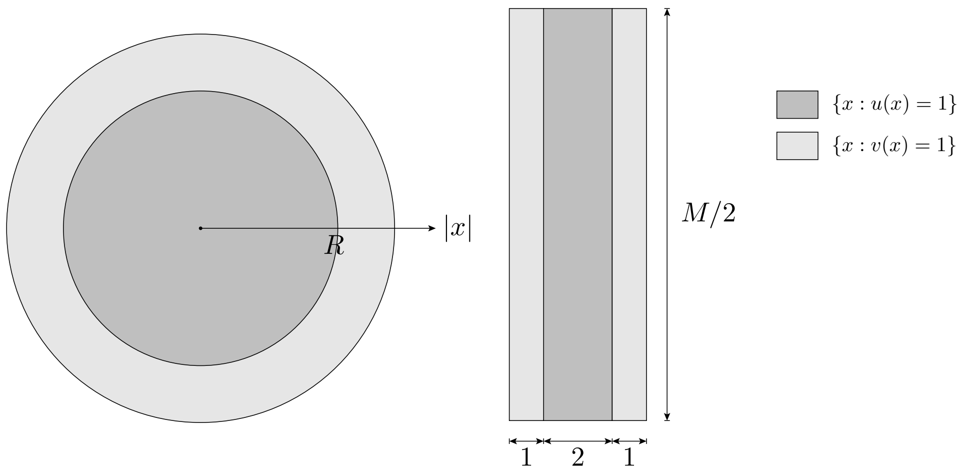

FIGURE 2. Disc and strip structures.

setup mass is transported (in the sense of the definition of $d_1(u, v)$) over a distance $O(R) = O(M^{1/2})$, so that

$$d_1(u, v) \sim MR \sim M^{3/2} \qquad \text{as } M \to \infty. \tag{2.1}$$

On the other hand, the interfacial length $\int |\nabla u|$ equals $2\pi R \sim M^{1/2}$, and the functional $\mathcal{F}_1$ picks up the larger of the two:

$$\mathcal{F}_1(u, v) = d_1(u, v) + \int |\nabla u| \sim M^{3/2}.$$

(2) For the **strip** we take the support of u to be a rectangle of width 2 and length $M/2$, flanked by two strips of half this width for $\operatorname{supp}(v)$ (Figure 2). The transport distance can now be taken constant and equal to 1, so that we find

$$\mathcal{F}_1(u, v) = M + M + 4 = 2M + 4.$$

For this choice of geometry $\mathcal{F}_1$ scales linearly with M in the limit $M \to \infty$.

Comparing the two we observe that the linear structure has lower energy, for large mass, than a spherical one. This is a first indication that $\mathcal{F}_1$ may favour partial localization.

With this example we can also illustrate an additional property. Let us take for $\operatorname{supp}(u)$ a strip of thickness t and length M/t, and recalculate the value of $\mathcal{F}_1$:

$$\mathcal{F}_1(u, v) = \left(\frac{t}{2} + \frac{2}{t}\right) M + 2t.$$

For large M this expression is dominated by the value of the prefactor $t/2 + 2/t$, suggesting two additional features:

(1) There is a preferred thickness, which is that value of t for which $t/2 + 2/t$ is minimal (i.e. $t = 2$);
(2) For the preferred thickness, the functional $\mathcal{F}_1$ equals $2M$ plus 'other terms'.

Although these statements only give suggestions, not proofs, for the case of general geometry, we shall see below that they both are true, and that these examples do

demonstrate the general behaviour. The 'other terms' in the example above are the single term $2t$, a term which is associated with the ends of the strip; we now turn to a different case, in which there are no loose ends, but instead the curvature of the structure creates an energy penalty.

2.2. **Fixed thickness ($\varepsilon = 1$), part 2: comparing line with ring structures.** In the previous section we argued that the energy $\mathcal{F}_1$ favours objects of thickness 2 and penalizes loose ends. We now consider *ring* structures, and we will see that also the *curvature* carries an energy penalty.

Let the supports of u and v be the ring structures of Fig. 3: the support of u is a single ring between circles of radii r_2 and r_3, and the support of v consists of two rings flanking $\mathrm{supp}(u)$.

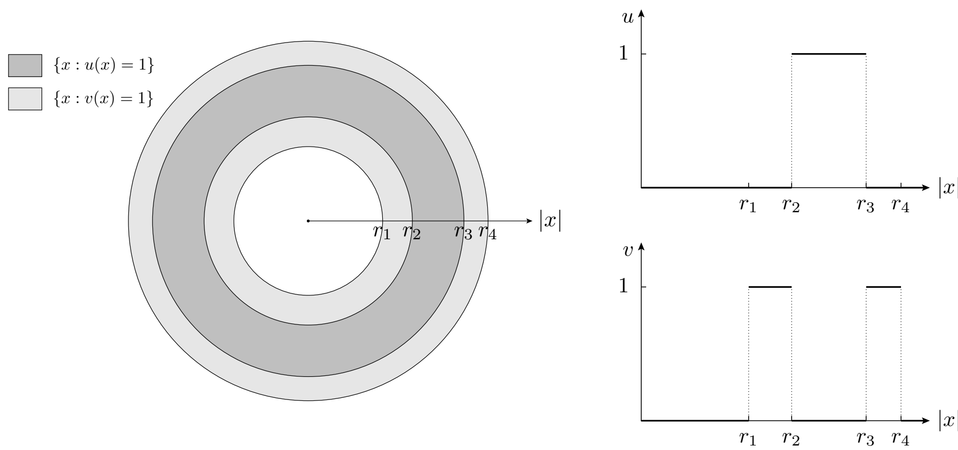

FIGURE 3. Ring structures; the pairs (u, v) on the right-hand figure indicate the values of u and v in that region in the plane.

We first do some direct optimization to reduce the number of degrees of freedom in this geometry. For fixed outer radii r_1 and r_4 one may optimize $\mathcal{F}_1$ over variations of r_2 and r_3 that respect the mass constraint $\int u = \int v$, and this optimization results in the values of r_2 and r_3 given by

$$r_2^2 = \tfrac{1}{2}(R^2 + r_1^2), \qquad \text{and} \qquad r_3^2 = \tfrac{1}{2}(R^2 + r_4^2) \tag{2.2}$$

in terms of the mean radius $R = \frac{1}{2}(r_1+r_4)$. The functional $\mathcal{F}_1$ can then be computed explicitly in terms of r_1 and r_4 (see Appendix B). The interesting quantity is actually the energy per unit mass, $\mathcal{F}_1/M$, as a function of the mean radius R and the thickness t of the structure,

$$M := \int u, \qquad t := \frac{r_4 - r_1}{2}.$$

Expanding $\mathcal{F}_1/M$ around $R = \infty$ and $t = 2$, we find

$$\frac{\mathcal{F}_1}{M}(R,t) = 2 + \frac{1}{4}(t-2)^2 + \frac{1}{4}R^{-2} + O(|t-2|^3 + R^{-3}), \tag{2.3}$$

see Appendix B.

Again we recognize a preference for structures of thickness $t = 2$; we now also observe a penalization of the curvature in the term $R^{-2}/4$. The main result of this paper indeed is to identify this curvature penalization for structures of arbitrary geometry, in the form of an elastica limit energy.

2.3. **Rescaling and renormalization.** In the previous sections we have seen that the functional $\mathcal{F}_\varepsilon$ at $\varepsilon = 1$ has a preference for structures of thickness 2. It is easy to see, by repeating the arguments above, that this becomes a preference for thickness 2ε in the general case. For instance, the development (2.3) generalizes to

$$\frac{\mathcal{F}_\varepsilon}{M}(R,t) = 2 + \frac{1}{4}(t/\varepsilon - 2)^2 + \frac{1}{4}\varepsilon^2 R^{-2} + O\big(|t/\varepsilon - 2|^3 + \varepsilon^3 R^{-3}\big).$$

This expression provides us with a recipe for identifying the bending energy in the limit $\varepsilon \to 0$. The curvature, which is equal to $1/R$ for this spherical geometry, enters the expression above as the term $\varepsilon^2 R^{-2}/4$. This suggests that the alternative functional

$$\mathcal{G}_\varepsilon(u,v) := \frac{1}{\varepsilon^2}\big[\mathcal{F}_\varepsilon(u,v) - 2M\big], \qquad \text{where} \qquad M = \int u = \int v,$$

may have a (Gamma-)limit similar to the Elastica functional. This suggestion is proved in Theorem 4.1.

3. Monge-Kantorovich distance, systems of curves and a generalized curve bending energy

In this section we introduce some basic definitions and concepts.

Definition 3.1. *Consider $u, v \in L^1(\mathbb{R}^2)$ with compact support in $\mathbb{R}^2$ satisfying the mass balance*

$$\int_{\mathbb{R}^2} u\, d\mathcal{L}^2 \;=\; \int_{\mathbb{R}^2} v\, d\mathcal{L}^2 \;=\; 1. \tag{3.1}$$

The Monge-Kantorovich distance $d_1(u,v)$ is defined as

$$d_1(u,v) \quad := \quad \min\Big(\int_{\mathbb{R}^2\times\mathbb{R}^2} |x-y|\, d\gamma(x,y)\Big) \tag{3.2}$$

where the minimum is taken over all Radon measures γ on $\mathbb{R}^2 \times \mathbb{R}^2$ with marginals *$u\mathcal{L}^2$ and $v\mathcal{L}^2$, that means γ satisfies*

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x)\, d\gamma(x,y) \quad = \quad \int_{\mathbb{R}^2} \varphi u\, d\mathcal{L}^2, \tag{3.3}$$

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \psi(y)\, d\gamma(x,y) \quad = \quad \int_{\mathbb{R}^2} \psi v\, d\mathcal{L}^2 \tag{3.4}$$

for all $\varphi, \psi \in C^0_c(\mathbb{R}^2)$.

The Monge-Kantorovich distance is characterized by the optimal mass transport problem described in section 5.

Remark 3.2. For $\mu, \nu \in \mathcal{P}_1$, where

$$\mathcal{P}_1(\mathbb{R}^2) \;:=\; \big\{\mu \text{ Radon measure on } \mathbb{R}^2 \;:\; \int_{\mathbb{R}^2} d\mu \;=\; 1,\; \int_{\mathbb{R}^2} |x|\, d\mu(x) \;<\; \infty\big\},$$

we can define the Monge-Kantorovich distance $d_1(\mu,\nu)$ analogously to (3.2), substituting $u\mathcal{L}^2$ and $v\mathcal{L}^2$ in (3.3), (3.4) by μ and ν. The minimum in (3.2) is always

attained and d_1 is a distance function on $\mathcal{P}_1$ [6]. Moreover d_1 is continuous in the sense that for $(\mu_k)_{k\in\mathbb{N}} \subset \mathcal{P}_1(\mathbb{R}^2), \mu \in \mathcal{P}_1(\mathbb{R}^2)$

$$d_1(\mu_k,\mu) \to 0 \qquad \Longleftrightarrow \qquad \begin{cases} \mu_k \to \mu \text{ as Radon measures and} \\ \int_{\mathbb{R}^2} |x|\,d\mu_k \to \int_{\mathbb{R}^2} |x|\,d\mu, \end{cases}$$

see [6, Theorem 3.2].

A family of related distances is given by the p-Wasserstein distances

$$d_p(\mu,\nu) \quad := \quad \min\Big(\int_{\mathbb{R}^2\times\mathbb{R}^2} |x-y|^p\,d\gamma(x,y)\Big)^{\frac{1}{p}}, \tag{3.5}$$

where the minimum is taken over all Radon measures γ on $\mathbb{R}^2\times\mathbb{R}^2$ with *marginals* μ and ν. The Monge-Kantorovich distance coincides with the 1-Wasserstein distance.

We will describe the limit structures in terms of *systems of closed $W^{2,2}$-curves*. The following definitions are mainly taken from [12].

Definition 3.3. *We call $\gamma : \mathbb{R} \to \mathbb{R}^2$ a closed $W^{2,2}$-curve if*

$$\begin{aligned} &\gamma \in W^{2,2}_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^2),\\ &\gamma'(r) \neq 0 \textit{ for all } r \in \mathbb{R},\\ &\gamma \textit{ is } L\textit{-periodic for some } 0 < L < \infty. \end{aligned}$$

If γ is L-periodic and injective on $[0,L)$ the curve γ is called simple. *The length $L(\gamma)$ of a closed curve γ is defined by*

$$L(\gamma) := \int_0^L |\gamma'(r)|\,dr,$$

where L is the the minimal period of γ.

A $W^{2,2}$-system of closed curves *is a finite collection of closed $W^{2,2}$-curves. We represent a system of curves by a* multiset $\{\gamma_i\}_{i=1,\dots,N}$, $N \in \mathbb{N}$, i.e. *a set in which repeated elements are counted with multiplicity.*

A system of curves $\Gamma = \{\gamma_i\}_{i=1,\dots,N}$ is called disjoint *if each γ_i is simple and $\gamma_i(\mathbb{R}) \cap \gamma_j(\mathbb{R}) = \emptyset$ for $i \neq j$. Γ has* no transversal crossings *if for any $1 \leq i,j \leq m$, $s_i, s_j \in \mathbb{R}$,*

$$\gamma_i(s_i) = \gamma_j(s_j) \textit{ implies that } \gamma_i'(s_i) \textit{ and } \gamma_j'(s_j) \textit{ are parallel.}$$

We define the support, multiplicity *and* total mass *of a system of curves $\Gamma = \{\gamma_i\}_{i=1,\dots,N}$ by*

$$\begin{aligned} \operatorname{supp}(\Gamma) &:= \bigcup_{i=1}^N \gamma_i(\mathbb{R}),\\ \theta(\Gamma,x) &:= \#\big\{(i,s) : \gamma_i(s) = x,\ 1 \leq i \leq N,\ 0 \leq s < L(\gamma_i)\big\},\\ |\Gamma| &:= \sum_{i=1}^N L(\gamma_i) \end{aligned}$$

and define a corresponding Radon measure μ_Γ on $\mathbb{R}^2$ to be the measure that satisfies

$$\int_{\mathbb{R}^2} \varphi\,d\mu_\Gamma \quad = \quad \sum_{i=1}^N \int_0^{L(\gamma_i)} \varphi(\gamma_i(s))\,|\gamma_i'(s)|\,ds \tag{3.6}$$

for all $\varphi \in C^0_c(\mathbb{R}^2)$.

Two systems of curves are identified if the corresponding Radon measures coincide.

Remark 3.4. We can represent a given system of closed curves Γ by a multiset $\Gamma = \{\gamma_i\}_{i=1,\dots,N}$, where for all $i = 1, \dots, N$

$$\gamma_i \text{ is one-periodic, with 1 being the smallest possible period,} \tag{3.7}$$

$$\gamma_i \text{ is parametrized proportional to arclength.} \tag{3.8}$$

We generalize the classical curve bending energy defined in (1.5) to $W^{2,2}$-systems of closed curves.

Definition 3.5. *Let Γ be a $W^{2,2}$-system of closed curves represented by $\Gamma = \{\gamma_i\}_{i=1,\dots,N}$ as in* (3.7)-(3.8) *and set $L_i = L(\gamma_i)$. Then we define*

$$\mathcal{W}(\Gamma) \quad := \quad \frac{1}{2}\sum_{i=1}^{N} L_i^{-3} \int_0^1 \gamma_i''(s)^2\, ds. \tag{3.9}$$

Remark 3.6. The support and total mass of a system of curves coincide with the support and total mass of the corresponding Radon measure [46]. The multiplicity functions of Γ and of μ_Γ coincide $\mathcal{H}^1$-almost everywhere on the support of Γ.
By Remark 3.8, Proposition 4.5, and Corollary 4.8 of [13], we obtain that Γ is a $W^{2,2}$-system of closed curves without transversal crossings if and only if μ_Γ is a *Hutchinson varifold* with weak mean curvature $\vec{H} \in L^2(\mu_\Gamma)$ such that in every point of $\mathrm{supp}(\Gamma)$ a unique tangent line exists. Moreover

$$\mathcal{W}(\Gamma) \quad = \quad \frac{1}{2}\int |\vec{H}|^2\, d\mu_\Gamma \tag{3.10}$$

holds.

4. Main results

To investigate the limit behaviour of the functionals $\mathcal{F}_\varepsilon$ as $\varepsilon \to 0$ we fix $M > 0$ and study the rescaled functionals

$$\mathcal{G}_\varepsilon(u, v) \quad := \quad \frac{1}{\varepsilon^2}\Big(\mathcal{F}_\varepsilon(u, v) - 2M\Big). \tag{4.1}$$

Theorem 4.1. *The curve bending energy $\mathcal{W}$ as defined in* (3.9) *is the Gamma-limit of the functionals $\mathcal{G}_\varepsilon$ in the following sense.*

1. *Let $(u_\varepsilon, v_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}^2) \times L^1(\mathbb{R}^2)$, $R > 0$ and a Radon measure μ on $\mathbb{R}^2$ be given with*

$$\mathrm{supp}(u_\varepsilon) \quad \subset \quad B_R(0) \quad \text{for all } \varepsilon > 0, \tag{4.2}$$

$$u_\varepsilon \mathcal{L}^2 \quad \to \quad \mu \quad \text{as Radon measures on } \mathbb{R}^2, \tag{4.3}$$

and

$$\liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon) \quad < \quad \infty. \tag{4.4}$$

Then there is a $W^{2,2}$-system of closed curves Γ such that

$$2\mu_\Gamma = \mu, \tag{4.5}$$

$$\mathrm{supp}(\Gamma) \text{ is bounded}, \tag{4.6}$$

$$\Gamma \text{ has no transversal crossings}, \tag{4.7}$$

$$2|\Gamma| \; = \; M, \tag{4.8}$$

and such that the liminf-estimate

$$\mathcal{W}(\Gamma) \quad \leq \quad \liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon(u_\varepsilon) \tag{4.9}$$

holds.

2. *Let* Γ *be a* $W^{2,2}$*-system of closed curves such that* (4.6)-(4.8) *holds. Then there exists a sequence* $(u_\varepsilon, v_\varepsilon)_{\varepsilon>0} \subset \mathcal{K}_\varepsilon$ *such that* (4.2) *holds for some* $R > 0$,

$$u_\varepsilon \mathcal{L}^2 \quad \to \quad 2\mu_\Gamma \qquad \text{as Radon measures on } \mathbb{R}^2, \tag{4.10}$$

and such that the limsup-estimate

$$\mathcal{W}(\Gamma) \quad \geq \quad \limsup_{\varepsilon\to 0} \mathcal{G}_\varepsilon(u_\varepsilon) \tag{4.11}$$

is satisfied.

Remark 4.2. Let X be the space of nonnegative Radon measures on $\mathbb{R}^2$. Define a concept of convergence on $X \times X$ by which

$$(\mu_i, \nu_i) \to (\mu, \nu) \quad \text{in } X \times X \quad \text{iff} \quad \begin{cases} \mu_i \to \mu & \text{as Radon measures on } \mathbb{R}^2 \text{ and} \\ \bigcup_{i\geq i_0} \operatorname{supp}(\mu_i) & \text{is bounded for some } i_0 \in \mathbb{N}\,. \end{cases}$$

Note that no condition is placed on ν_i. We now define

$$\mathcal{G}_\varepsilon(\mu,\nu) \quad := \quad \begin{cases} \mathcal{G}_\varepsilon(u,v) & \text{if } \mu = u\mathcal{L}^2,\ \nu = v\mathcal{L}^2 \text{ for some } (u,v) \in \mathcal{K}_\varepsilon, \\ \infty & \text{otherwise} \end{cases}$$

and

$$\mathcal{G}_0(\mu,\nu) \quad := \quad \begin{cases} \mathcal{W}(\Gamma) & \begin{cases} \text{if } \mu = \nu \text{ is given by a } W^{2,2}\text{-system of closed curves} \\ \Gamma = \{\gamma_i\}_{i=1,\dots,N} \text{ satisfying (4.5)-(4.8)} \end{cases} \\ \infty & \text{otherwise.} \end{cases}$$

Then Theorem 4.1 can be rephrased as

$$\lim_{\varepsilon\to 0} \mathcal{G}_\varepsilon \quad = \quad \mathcal{G}_0 \tag{4.12}$$

in the sense of Gamma-convergence with respect to the topology of $X \times X$ defined above.

Remark 4.3. Implicit in the formulation of Theorem 4.1 is the following (contrapositive) statement: if a sequence $(u_\varepsilon, v_\varepsilon)_{\varepsilon>0} \subset \mathcal{K}_\varepsilon$ converges to a measure that *cannot* be represented by a system of curves satisfying (4.5)–(4.8), then $\mathcal{G}_\varepsilon$ is unbounded along this sequence. Put differently, each of the following is penalized in $\mathcal{F}_\varepsilon$ at an order lower (*i.e.* larger) than ε^2:

(1) a fracture in the limit structure, occurring as a nonclosed curve;
(2) a total mass $|\Gamma|$ that deviates from $M/2$;
(3) a non-even multiplicity.

In the Introduction we conjectured the asymptotic development (1.6). Theorem 4.1 justifies the zeroth and second order of this development in the following way: if $F_\varepsilon - 2M$ is of order $O(\varepsilon^2)$, then the three types of degeneracy above can not occur, and the dominant term is given in the limit by $\varepsilon^2\mathcal{W}$.

Remark 4.4. By the definition of the set $\mathcal{K}_\varepsilon$ and the functionals $\mathcal{F}_\varepsilon$ in (1.1), (1.2) it is clear that for a sequence $(u_\varepsilon, v_\varepsilon)_{\varepsilon>0}$ along which $\mathcal{G}_\varepsilon$ is bounded the supports of u_ε and v_ε necessarily concentrate as $\varepsilon \to 0$. The crucial conclusions are (a) that the concentration is on curves rather than on points or other sets and (b) that the limit measure can be constructed as a sum of curves, each with density two. The latter property implies a uniform thickness of structures with bounded energy and paraphrases the stretching resistance of lipid bilayers.

Remark 4.5. We also obtain a compactness result for sequences with bounded $\mathcal{G}_\varepsilon$: Let $(u_\varepsilon, v_\varepsilon)_{\varepsilon>0} \subset L^1(\mathbb{R}^2) \times L^1(\mathbb{R}^2)$ satisfy (4.2), (4.4). Then there exists a Radon measure μ on $\mathbb{R}^2$ such that (4.3) holds for a subsequence $\varepsilon \to 0$. Theorem 4.1 then implies that there is a $W^{2,2}$-system Γ of closed curves such that the conclusions (4.5)-(4.9) hold.

5. The Mass Transport Problem

In this section we recall the classical Monge problem of optimal mass transport and review some results that we will use later.

Definition 5.1. *Fix two non-negative Borel functions $u, v \in L^1(\mathbb{R}^2)$ with compact support satisfying the mass balance* (3.1). *By $\mathcal{A}(u,v)$ we denote the set of all Borel maps $S : \mathbb{R}^2 \to \mathbb{R}^2$* pushing u forward to v, *that is*

$$\int \eta(S(x))u(x)\,dx \quad = \quad \int_{\mathbb{R}^2} \eta(y)v(y)\,dy \tag{5.1}$$

holds for all $\eta \in C^0(\mathbb{R}^2)$.

Optimal mass transport. *For u, v as in Definition 5.1 minimize the* transport cost $I : \mathcal{A}(u,v) \to \mathbb{R}$,

$$I(S) \quad := \quad \int_\Omega |S(x) - x| u(x)\,dx. \tag{5.2}$$

The following dual formulation is due to Kantorovich [33].

Optimal Kantorovich potential. *For u, v as in Definition 5.1 maximize K : $\mathrm{Lip}_1(\mathbb{R}^2) \to \mathbb{R}$,*

$$K(\phi) \quad := \quad \int_\Omega \phi(x)(u - v)(x)dx, \tag{5.3}$$

where $\mathrm{Lip}_1(\mathbb{R}^2)$ *denotes the space of Lipschitz functions on $\mathbb{R}^2$ with Lipschitz constant not larger than* 1.

There is a vast literature on the optimal mass transportation problem and an impressive number of applications, see for example [25, 48, 16, 5, 50, 32, 40]. We only list a few results which we will use later.

Theorem 5.2 ([16],[27]). *Let u, v be given as in Definition 5.1.*
1. There exists an optimal transport map $S \in \mathcal{A}(u,v)$.
2. There exists an optimal Kantorovich potential $\phi \in \mathrm{Lip}_1(\mathbb{R}^2)$.
3. The identities

$$d_1(u,v) \;=\; I(S) \;=\; K(\phi)$$

hold.

4. Every optimal transport map S and every optimal Kantorovich potential ϕ satisfy

$$\phi(x) - \phi(S(x)) \quad = \quad |x - S(x)| \qquad \text{for almost all } x \in \operatorname{supp}(u). \tag{5.4}$$

The optimal transport map and the optimal Kantorovich potential are in general not unique. We can choose S and ϕ enjoying some additional properties.

Proposition 5.3 ([16, 27])**.** *There exists an optimal transport map $S \in \mathcal{A}(u,v)$ and an optimal Kantorovich potential ϕ such that*

$$\phi(x) \quad = \quad \min_{y \in \operatorname{supp}(v)} \big(\phi(y) + |x-y|\big) \qquad \text{for any } x \in \operatorname{supp}(u), \tag{5.5}$$

$$\phi(y) \quad = \quad \max_{x \in \operatorname{supp}(u)} \big(\phi(x) - |x-y|\big) \qquad \text{for any } y \in \operatorname{supp}(v), \tag{5.6}$$

and such that S is the unique monotone transport map in the sense of [27]*,*

$$\frac{x_1 - x_2}{|x_1 - x_2|} + \frac{S(x_1) - S(x_2)}{|S(x_1) - S(x_2)|} \neq 0 \qquad \text{for all } x_1 \neq x_2 \in \mathbb{R}^2 \text{ with } S(x_1) \neq S(x_2).$$

We will extensively use the fact that, by (5.4) the optimal transport is organized along *transport rays* which are defined as follows.

Definition 5.4 ([16])**.** *Let u,v be as in Definition 5.1 and let $\phi \in \operatorname{Lip}_1(\mathbb{R}^2)$ be the optimal transport map as in Proposition 5.3. A transport ray is a line segment in $\mathbb{R}^2$ with endpoints $a,b \in \mathbb{R}^2$ such that ϕ has slope one on that segment and a,b are maximal, that is*

$$\begin{gathered} a \in \operatorname{supp}(u),\ b \in \operatorname{supp}(v), \quad a \neq b, \\ \phi(a) - \phi(b) \ = \ |a-b| \\ |\phi(a + t(a-b)) - \phi(b)| \ < \ |a + t(a-b) - b| \qquad \text{for all } t > 0, \\ |\phi(b + t(b-a)) - \phi(a)| \ < \ |b + t(b-a) - a| \qquad \text{for all } t > 0. \end{gathered}$$

We define the transport set $\mathcal{T}$ to consist of all points which lie in the (relative) interior of some transport ray and define $\mathcal{E}$ to be the set of all endpoints of rays.

Some important properties of transport rays are given in the next proposition.

Proposition 5.5 ([16])**.**
1. Two rays can only intersect in a common endpoint.
2. The endpoints $\mathcal{E}$ form a Borel set of measure zero.
3. If z lies in the interior of a ray with endpoints $a \in \operatorname{supp}(u), b \in \operatorname{supp}(v)$ then ϕ is differentiable in z with $\nabla\phi(z) = (a-b)/|a-b|$.

In Section 7.1 we will use the transport rays to parametrize the support of u and to compute the Monge-Kantorovich distance between u and v.

6. Overview of the proof of Theorem 4.1

6.1. **The lower bound.** The heart of the proof of the lower bound (4.9) is a parametrization of $\operatorname{supp}(u) \cup \operatorname{supp}(v)$ that allows us to rewrite the functionals $\mathcal{F}_\varepsilon$ and $\mathcal{G}_\varepsilon$ in terms of the geometry of $\operatorname{supp}(u)$ and $\operatorname{supp}(v)$ and the structure of the transport rays.

The first step in the proof of (4.9) is to pass to functions u and v with smooth boundaries (Proposition (7.1)) so that $\partial \operatorname{supp}(u)$ can be represented by a collection $\{\gamma_i\}$ of smooth curves. For the discussion in this section we will pretend that there

is only one curve $\gamma : [0, L) \to \mathbb{R}^2$, which is closed and which we parametrize by arclength.

The properties of the Monge-Kantorovich distance d_1 give that almost every point $x \in \operatorname{supp}(u) \cup \operatorname{supp}(v)$ lies in the interior of exactly one transport ray; the set $\operatorname{supp}(u) \cup \operatorname{supp}(v)$ can therefore be written, up to a null set, as the disjoint union of transport rays.

In addition, along each ray mass is transported 'from u to v', implying that each ray intersects $\partial \operatorname{supp}(u)$. We therefore can parametrize the collection of rays by their intersections with $\partial \operatorname{supp}(u)$.

The ray direction of the ray $\mathcal{R}(s)$ that passes through $\gamma(s)$ defines a unit length vector $\theta(s)$; we now introduce a parametrization ψ by

$$\psi(s,t) := \gamma(s) + t\theta(s).$$

In terms of such a parametrization the first term of $\mathcal{F}_\varepsilon$ has a simple description as the length of the curve γ,

$$\varepsilon \int |\nabla u| = \int_0^L 1\, ds = L.$$

To rewrite $d_1(u,v)$ we use the fact that transport takes place along transport rays; when restricted to a single ray the transport problem becomes one-dimensional, and even explicitly solvable.

The central estimate in this part of the proof is the following lower bound:

$$d_1(u,v) \geq \int_0^L \left[\frac{\varepsilon}{\sin\beta(s)} M(s)^2 + \frac{\varepsilon^3}{4\sin^5\beta(s)} \alpha'(s)^2 M(s)^4 \right] ds. \tag{6.1}$$

In this inequality the geometry of $\operatorname{supp}(u)$ is characterized by functions α, β, and M, which are illustrated in Figure 4. The functions α and β are angles: $\alpha(s)$ is the

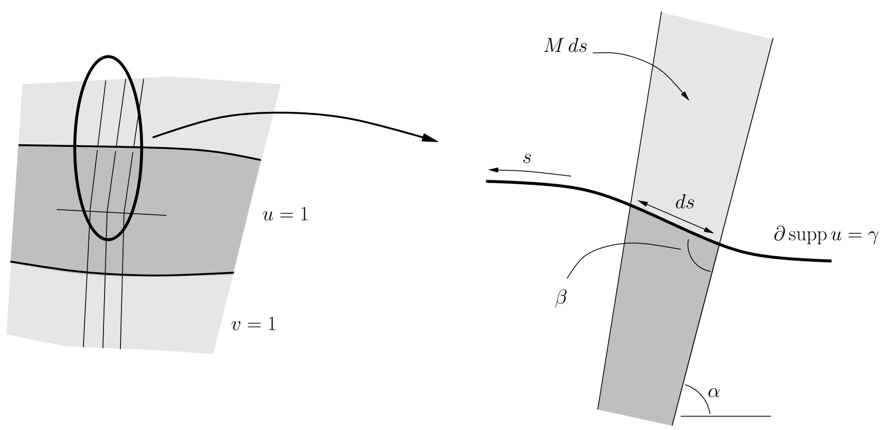

FIGURE 4. The parametrization of $\operatorname{supp}(u) \cup \operatorname{supp}(v)$. On the left six transport rays are drawn over the two supports; on the right the functions α, β, and M are indicated. At a boundary point $\gamma(s)$, $\alpha(s)$ is the angle of the transport ray through that point, and $\beta(s)$ is the angle between γ and the ray. Finally, $M(s)ds$ is the amount of mass contained between the rays at positions s and $s + ds$.

angle between the ray direction $\theta(s)$ and a reference direction and $\beta(s)$ is the angle between $\theta(s)$ and the tangent $\gamma'(s)$ to γ. The function $M(\cdot)$, finally, measures the amount of mass supported on a ray. The relation between the function $M(\cdot)$ and the scalar M in (1.2) is

$$M = \int_0^L M(s)\,ds.$$

With the estimate (6.1), the main result can readily be appreciated. Let a sequence $(u_\varepsilon, v_\varepsilon)$ be such that $\mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon)$ is bounded as $\varepsilon \to 0$, and let α_ε, β_ε, and M_ε be the associated geometric quantities. With the inequality above,

$$\begin{aligned}\mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon) &= \frac{\mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) - 2M_\varepsilon}{\varepsilon^2}\\ &\geq \int_0^L \Big[\frac{1}{\varepsilon^2} - \frac{2M_\varepsilon(s)}{\varepsilon^2} + \frac{M_\varepsilon(s)^2}{\varepsilon^2 \sin\beta_\varepsilon(s)} + \frac{\alpha_\varepsilon'(s)^2 M_\varepsilon(s)^4}{4\sin^5\beta_\varepsilon(s)}\Big]ds\\ &= \int_0^L \left[\frac{1}{\varepsilon^2}\big(1 - M_\varepsilon(s)\big)^2 + \frac{1}{\varepsilon^2}\left(\frac{1}{\sin\beta_\varepsilon(s)} - 1\right)M_\varepsilon(s)^2 + \frac{\alpha_\varepsilon'(s)^2 M_\varepsilon(s)^4}{4\sin^5\beta_\varepsilon(s)}\right]ds\end{aligned} \tag{6.2}$$

Note that each of the three terms above is non-negative. If $\mathcal{G}_\varepsilon(u, v)$ is bounded, then one concludes that as $\varepsilon \to 0$ the mass $M_\varepsilon(s)$ tends to one for almost all s. Similarly, β_ε converges to $\pi/2$, implying that the ray angle α_ε converges to the angle of the normal to γ_ε, and in a weak sense therefore α_ε' converges to the curvature $\kappa_\varepsilon = \gamma_\varepsilon''$. The inequality above suggests a $W^{2,2}_{\mathrm{loc}}$ bound for curves γ_ε with the last term approximating

$$\frac{1}{4}\int_0^L \kappa_\varepsilon(s)^2\,ds$$

This integral is the bending energy that we expect in the limit.

Based on the inequality (6.2) we show that the boundary curves of $\operatorname{supp}(u_\varepsilon)$ are compact, that the limit is given by a $W^{2,2}$ system of closed curves as in (4.5)-(4.8) and that the lim-inf estimate is satisfied. We finally conclude that the limit of the mass distributions u_ε is identical to the limit of the boundary curves.

The main reasons why the proof in Section 7 is more involved than this brief explanation are related to the gaps in the reasoning above:

- A ray can intersect $\partial \operatorname{supp}(u)$ multiple times, and care must be taken to ensure that the parametrization is a bijection.
- The ray direction θ is not necessarily smooth; to be precise, when the ray length tends to zero (which is equivalent to $M(\cdot)$ vanishing), θ may vary wildly.
- Similarly, when $M(\cdot)$ vanishes, the L^2-estimate on α' in (6.2) degenerates, resulting in a compactness problem for the boundary curves.

6.2. **The upper bound.** With the machinery of the lower bound in place, and with the insight that is provided, the upper bound becomes a relatively simple construction. Around a given limit curve tubular neighbourhoods are constructed for the supports of u and v. The thickness of these neigbourhoods can be chosen just right, and the calculation that leads to (6.1) now gives an exact value for $\mathcal{G}_\varepsilon$.

The main remaining difficulty of the proof is the fact that a $W^{2,2}$-curve need not have a non-self-intersecting tubular neighbourhood of any thickness, and therefore an approximation argument is needed.

7. Proof of the lim-inf estimate

In this section we prove the first part of Theorem 4.1. The main idea is to use the optimal Kantorovich potential of the mass transport problem of u_ε to v_ε to construct a parametrization of the support of u_ε and v_ε and to derive a lower bound for the functionals $\mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon)$. This lower bound yields the compactness of the systems of curves which describe the boundary of the support of u_ε as well as the desired lim-inf estimate.

We first show that we can restrict ourselves to a class of 'generic data'.

Proposition 7.1. *It is sufficient to prove the* liminf *part of Theorem 4.1 under the additional assumptions that*

$$M = 1, \quad \textit{that is} \int u_\varepsilon = \int v_\varepsilon = 1 \quad \textit{for } (u_\varepsilon, v_\varepsilon) \in \mathcal{K}_\varepsilon, \tag{7.1}$$

$$\sup_{\varepsilon>0} \mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon) \le \Lambda \quad \textit{for some} \quad 0 < \Lambda < \infty, \tag{7.2}$$

$$(u_\varepsilon, v_\varepsilon) \in \mathcal{K}_\varepsilon \quad \textit{for all } \varepsilon > 0, \tag{7.3}$$

$$\lim_{\varepsilon\to 0} \mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon) = \liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon) \quad \textit{exists,} \tag{7.4}$$

$$\partial\, \mathrm{supp}(u_\varepsilon) \textit{ is given by a disjoint system of smooth, simple curves } \Gamma_\varepsilon. \tag{7.5}$$

Proof. To prove that we can restrict ourselves to $M = 1$ we do a spatial rescaling and set

$$\tilde\varepsilon := \frac{\varepsilon}{M}, \quad \tilde u_{\tilde\varepsilon}(x) := M u_\varepsilon(Mx), \quad \tilde v_{\tilde\varepsilon}(x) := M v_\varepsilon(Mx).$$

We then obtain

$$\int_{\mathbb{R}^2} \tilde u_{\tilde\varepsilon} = \int_{\mathbb{R}^2} \tilde v_{\tilde\varepsilon} = 1, \qquad\qquad \tilde\varepsilon \int |\nabla \tilde u_\varepsilon| = \frac{\varepsilon}{M} \int |\nabla u_\varepsilon|. \tag{7.6}$$

Moreover, if S is the optimal transport map from u_ε to v_ε, then $\tilde S(x) := M^{-1}S(Mx)$ is the optimal transport map from $\tilde u_{\tilde\varepsilon}$ to $\tilde v_{\tilde\varepsilon}$ and

$$d_1(\tilde u_{\tilde\varepsilon}, \tilde v_{\tilde\varepsilon}) = M^{-2} d_1(u_\varepsilon, v_\varepsilon). \tag{7.7}$$

We obtain from (7.6), (7.7) that

$$\begin{aligned}
\mathcal{G}_{\tilde\varepsilon}(\tilde u_{\tilde\varepsilon}, \tilde v_{\tilde\varepsilon}) &= \frac{1}{\tilde\varepsilon^2}\Big(\frac{1}{\tilde\varepsilon} d_1(\tilde u_{\tilde\varepsilon}, \tilde v_{\tilde\varepsilon}) + \tilde\varepsilon \int |\nabla \tilde u_{\tilde\varepsilon}| - 2\Big) \\
&= \frac{M^2}{\varepsilon^2}\Big(\frac{1}{M\varepsilon} d_1(u_\varepsilon, v_\varepsilon) + \frac{\varepsilon}{M}\int |\nabla u_\varepsilon| - \frac{2}{M}\int u_\varepsilon\Big) \\
&= M\mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon).
\end{aligned} \tag{7.8}$$

We observe that $(\tilde u_{\tilde\varepsilon}, \tilde v_{\tilde\varepsilon}) \in \mathcal{K}_{\tilde\varepsilon}$ iff $(u_\varepsilon, v_\varepsilon) \in \mathcal{K}_\varepsilon$ and that $(\tilde u_{\tilde\varepsilon})_{\tilde\varepsilon>0}$ satisfies (4.2), (4.3) for $\tilde R := R/M$ and $\tilde\mu$ defined by

$$\int \eta(x)\, d\tilde\mu(x) = \int \frac{1}{M}\eta\Big(\frac{x}{M}\Big)\, d\mu(x) \quad \text{for } \eta \in C_c^0(\mathbb{R}^2). \tag{7.9}$$

If Theorem 4.1 holds for $M = 1$ then there exists a $W^{2,2}$-system of closed curves $\tilde\Gamma$ with

$$2\mu_{\tilde\Gamma} = \tilde\mu, \qquad 2|\tilde\Gamma| = 1, \tag{7.10}$$

such that (4.6), (4.7) holds and

$$\mathcal{W}(\tilde\Gamma) \leq \liminf_{\tilde\varepsilon\to 0} \mathcal{G}_{\tilde\varepsilon}(\tilde u_{\tilde\varepsilon}, \tilde v_{\tilde\varepsilon}). \tag{7.11}$$

We deduce from (7.9) and (7.10) that μ is given as system of closed curves: Let $\tilde\Gamma = \{\tilde\gamma_i\}_{i=1,\dots,N}$ with

$$\tilde\gamma_i : [0, \tilde L_i) \to \mathbb{R}^2, \qquad |\tilde\gamma_i'| = 1$$

and define a system of curves $\Gamma = \{\gamma_i\}_{i=1,\dots,N}$ by

$$\begin{aligned} \gamma_i : [0, L_i) \to \mathbb{R}^2, \quad L_i := M\tilde L_i, \\ \gamma_i(s) := M\tilde\gamma\Big(\frac{s}{M}\Big) \quad \text{ for } s \in [0, L_i). \end{aligned}$$

Then $|\gamma_i'| = 1$ and Γ satisfies (4.6), (4.7). We obtain from (7.9) that

$$\begin{aligned} \int \eta\, d\mu &= M \int \eta(Mx)\, d\tilde\mu(x) = 2M \sum_{i=1}^m \int_0^{\tilde L_i} \eta(M\tilde\gamma_i(s))\, ds \\ &= 2\sum_{i=1}^m \int_0^{L_i} \eta(\gamma_i(s))\, ds = 2\int \eta\, d\mu_\Gamma, \end{aligned}$$

which yields $\mu = 2\mu_\Gamma$ and, by taking $\eta = 1$ inside $B_R(0)$, that (4.8) holds. Finally we deduce from (7.9)

$$\begin{aligned} \mathcal{W}(\Gamma) &= \frac{1}{2}\sum_{i=1}^m \int_0^{L_i} |\gamma_i''(s)|^2\, ds \\ &= \frac{1}{2}\sum_{i=1}^m \int_0^{L_i} \frac{1}{M^2}\Big|\tilde\gamma_i''\Big(\frac{s}{M}\Big)\Big|^2 ds = \frac{1}{2}\sum_{i=1}^m \int_0^{\tilde L_i} \frac{1}{M}|\tilde\gamma_i''(s)|^2\, ds \\ &= \frac{1}{M}\mathcal{W}(\tilde\Gamma) \end{aligned}$$

and by (7.8), (7.11) this yields that (4.9) holds. Therefore we have shown that it is sufficient to prove the lim-inf part of Theorem 4.1 for $M = 1$.

Eventually restricting ourselves to a subsequence $\varepsilon \to 0$ we can assume (7.4). The existence of a $0 < \Lambda < \infty$ such that (7.2) holds follows from (4.4). By the definition of $\mathcal{G}_\varepsilon$ this in particular implies (7.3).

It remains to show that it is sufficient to consider functions u_ε such that $\partial\,\mathrm{supp}(u_\varepsilon)$ is smooth. We already have seen that we can assume that $(u_\varepsilon, v_\varepsilon) \in \mathcal{K}_\varepsilon$. This implies that $\mathrm{supp}(u_\varepsilon)$ is a set of finite perimeter. Moreover $\mathrm{supp}(u_\varepsilon) \subset B_R(0)$ and by [7, Theorem 3.42] there exists a sequence of open sets $(E_k)_{k\in\mathbb{N}}$ with smooth boundary such that

$$\begin{aligned} \mathcal{X}_{E_k} &\to \varepsilon u_\varepsilon \quad \text{ in } L^1(\mathbb{R}^2), && (7.12) \\ |\nabla\mathcal{X}_{E_k}|(\mathbb{R}^2) &\to \varepsilon|\nabla u_\varepsilon|(\mathbb{R}^2) && (7.13) \\ E_k &\subset B_{2R}(0). && (7.14) \end{aligned}$$

We set

$$r_k \ := \ \varepsilon^{\frac{1}{2}}|E_k|^{-\frac{1}{2}} \tag{7.15}$$

and define functions $\hat{u}_k$,

$$\hat{u}_k(x) \ := \ \frac{1}{\varepsilon}\mathcal{X}_{E_k}(r_k^{-1}x). \tag{7.16}$$

This yields

$$\int_{\mathbb{R}^2} \hat{u}_k \ = \ \frac{r_k^2}{\varepsilon}|E_k| \ = \ 1 \tag{7.17}$$

and

$$\int_{\mathbb{R}^2} |\nabla \hat{u}_k| \ = \ \frac{r_k}{\varepsilon}\int_{\mathbb{R}^2} |\nabla E_k|. \tag{7.18}$$

Moreover, from (7.12), (7.15) and $|\operatorname{supp}(u_\varepsilon)| = \varepsilon$ we observe that

$$r_k \ \to \ 1 \quad \text{ as } k \to \infty. \tag{7.19}$$

We compute that

$$\begin{aligned}
\int_{\mathbb{R}^2} |\hat{u}_k - u_\varepsilon| \ &= \ \frac{1}{\varepsilon}\int_{\mathbb{R}^2} \Big|\mathcal{X}_{E_k}(r_k^{-1}x) - \varepsilon u_\varepsilon(x)\Big|\,dx \\
&\leq \ \frac{1}{\varepsilon}\int_{\mathbb{R}^2} \big|\mathcal{X}_{E_k}(r_k^{-1}x) - \varepsilon u_\varepsilon(r_k^{-1}x)\big|\,dx \\
&\quad +\frac{1}{\varepsilon}\int_{\mathbb{R}^2} \big|\varepsilon u_\varepsilon(r_k^{-1}x) - \varepsilon u_\varepsilon(x)\big|\,dx \\
&= \ \frac{r_k^2}{\varepsilon}\int_{\mathbb{R}^2} \big|\mathcal{X}_{E_k} - \varepsilon u_\varepsilon(x)\big|\,dx + \int_{\mathbb{R}^2} \big|u_\varepsilon(r_k^{-1}x) - u_\varepsilon(x)\big|\,dx.
\end{aligned}$$

Since the right-hand side converges to zero as $k \to \infty$ by (7.12), (7.19) we deduce that

$$\hat{u}_k \quad \to \quad u_\varepsilon \quad \text{ in } L^1(\mathbb{R}^2) \tag{7.20}$$

as $k \to \infty$. Equations (7.13), (7.18) and (7.19) yield that

$$\int_{\mathbb{R}^2} |\nabla \hat{u}_k| \quad \to \quad \int_{\mathbb{R}^2} |\nabla u_\varepsilon| \quad \text{ as } k \to \infty. \tag{7.21}$$

We construct approximations $\hat{v}_k$ of v_ε by setting

$$\hat{v}_k(x) \ := \ v_\varepsilon(x)\big(1 - \varepsilon\hat{u}_k(x)\big) + w_k(x), \tag{7.22}$$

for a suitable $w_k \in L^1\big(B_{2R}(0), \{0, \varepsilon^{-1}\}\big)$ with

$$\int_{\mathbb{R}^2} (w_k - \varepsilon v_\varepsilon \hat{u}_k) \ = \ 0. \tag{7.23}$$

We then deduce from (7.22), (7.23) that $(\hat{u}_k, \hat{v}_k) \in \mathcal{K}_\varepsilon$. It follows from (7.20), (7.23) and $u_\varepsilon v_\varepsilon = 0$ that $w_k \to 0$ as $k \to \infty$. Together with (7.22) we obtain that

$$\lim_{k\to\infty} \hat{v}_k \ = \ v_\varepsilon \quad \text{ in } L^1(\mathbb{R}^2). \tag{7.24}$$

By (7.14), (7.20) and (7.24) we deduce from Remark 3.2 the continuity of the Monge-Kantorovich distance term in $\mathcal{F}_\varepsilon(\hat{u}_k, \hat{v}_k)$. Together with (7.21) this yields

$$\mathcal{F}_\varepsilon(\hat{u}_k, \hat{v}_k) \quad \to \quad \mathcal{F}_\varepsilon(u_\varepsilon, v_\varepsilon) \quad \text{ as } k \to \infty.$$

Therefore we can choose $k(\varepsilon) \in \mathbb{N}$ such that for $\tilde{u}_\varepsilon := \hat{u}_{k(\varepsilon)}$, $\tilde{v}_\varepsilon := \hat{v}_{k(\varepsilon)}$ satisfy the estimate

$$\|u_\varepsilon - \tilde{u}_\varepsilon\|_{L^1(\mathbb{R}^2)} + |\mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon) - \mathcal{G}_\varepsilon(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)| \quad \leq \quad \varepsilon.$$

This yields a sequence $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)_{\varepsilon>0}$ that satisfies (7.5) and has the same limit as $(u_\varepsilon, v_\varepsilon)_{\varepsilon>0}$ in (4.3), (4.9). ☐

From now on, for the rest of this section, we assume that (7.1)-(7.5) hold.

7.1. **Parametrization by rays.** Due to (7.5) the boundary $\partial\,\mathrm{supp}(u_\varepsilon)$ is smooth. We are free to choose a suitable representation.

Remark 7.2. We can represent $\partial\,\mathrm{supp}(u_\varepsilon)$ by a disjoint system Γ_ε of closed curves such that for all $\gamma \in \Gamma_\varepsilon$

$$\begin{aligned} &\gamma \text{ is parametrized by arclength}, |\gamma'| = 1, \\ &\mathrm{supp}(u_\varepsilon) \text{ is `on the left hand side' of } \gamma. \end{aligned} \tag{7.25}$$

In particular γ is $L(\gamma)$-periodic and

$$\det(\gamma'(s), \nu(s)) \;\leq\; 0,$$

where $\nu(s)$ denotes the outward unit normal of $\mathrm{supp}(u_\varepsilon)$ at $\gamma(s)$.

Let $\phi \in \mathrm{Lip}_1(\mathbb{R}^2)$ be an optimal Kantorovich potential for the mass transport from u_ε to v_ε as in Proposition 5.3, with $\mathcal{T}$ being the set of transport rays as in Definition 5.4. Recall that ϕ is differentiable, with $|\nabla\phi| = 1$, in the relative interior of any ray.

Definition 7.3 (Parametrization by rays)**.** *For $\gamma \in \Gamma_\varepsilon$ we define*

(1) a set E of 'inner points' with respect to the transport set $\mathcal{T}$,

$$E \quad := \quad \{s \in \mathbb{R} : \gamma(s) \in \mathcal{T}\},$$

(2) a direction field

$$\theta : E \to \mathcal{S}^1, \qquad \theta(s) := \nabla\phi(\gamma(s)),$$

(3) the positive and negative total ray length $L^+, L^- : E \to \mathbb{R}$,

$$L^+(s) \quad := \quad \sup\{t > 0 : \phi(\gamma(s) + t\theta(s)) - \phi(\gamma(s)) \;=\; t\}, \tag{7.26}$$

$$L^-(s) \quad := \quad \inf\{t < 0 : \phi(\gamma(s) + t\theta(s)) - \phi(\gamma(s)) \;=\; t\}, \tag{7.27}$$

(4) the effective *positive ray length $l^+ : E \to \mathbb{R}$,*

$$l^+(s) := \sup\,\{t \in [0, L^+(s)] : \gamma(s) + \tau\theta(s) \in \mathrm{Int}\big(\mathrm{supp}(u_\varepsilon)\big) \textit{ for all } 0 < \tau < t\}. \tag{7.28}$$

Finally we define a map ψ which will serve as a parametrization of $\mathrm{supp}(u_\varepsilon) \cup \mathrm{supp}(v_\varepsilon)$ *by*

$$\psi : \big[0, L(\gamma)\big) \times \mathbb{R} \;\to\; \mathbb{R}^2, \quad \psi(s,t) \;:=\; \gamma(s) + t\theta(s). \tag{7.29}$$

Remark 7.4. All objects defined above are properties of γ even if we do not denote this dependence explicitely. When dealing with a collection of curves $\{\gamma_i : i = 1, ..., N\}$ or $\{\gamma_{\varepsilon,i} : \varepsilon > 0, i = 1, ..., N(\varepsilon)\}$ then E_i, $l^+_{\varepsilon,i}$ etc. refer to the objects defined for the corresponding curves.

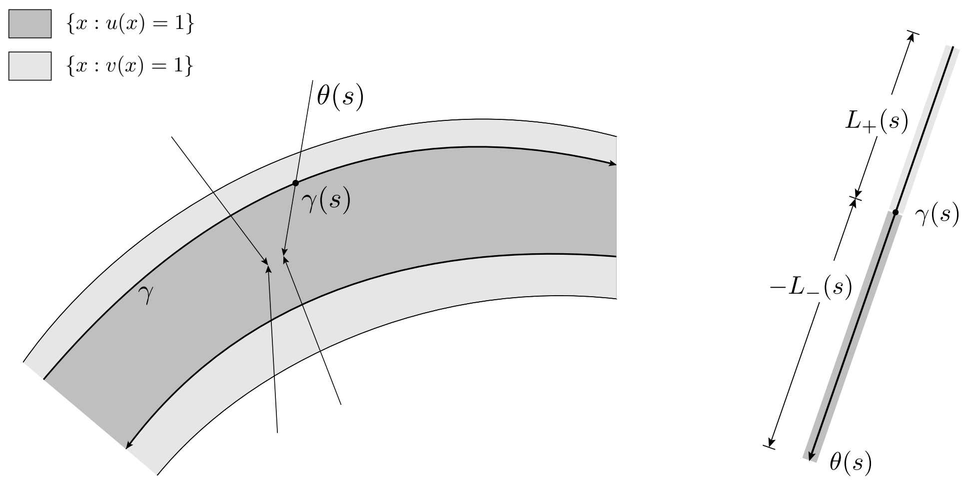

FIGURE 5. The parametrization in a 'simple' situation. Here $l^+(s) = L^+(s)$ coincide.

See the Figures 5 and 6 for an illustration of the parametrization defined above. The *effective ray length* is introduced to obtain the injectivity of the parametrization in the case that a ray crosses several times the boundary of $\mathrm{supp}(u_\varepsilon)$ (as it is the case in the situation depicted in Figure 6). The set $\{\gamma(s) : l^+(s) > 0\}$ represents the points of the boundary where mass is transported in the 'right direction'.

We will first prove some results which are analogous to those in [16].

Lemma 7.5. *The ray direction $\theta : E \to \mathbb{R}^2$ is continuous. The positive, negative, and effective positive ray lengths $L^+, L^-, l^+ : E \to \mathbb{R}$ are measurable.*

Proof. Consider a sequence $(s_k)_{k\in\mathbb{N}} \subset E$, $s \in E$, with $s_k \to s$. Choose $\tilde{t}_k \le t_k$ such that

$$L^+(s_k) - \frac{1}{k} < t_k < L^+(s_k), \tag{7.30}$$

$$l^+(s_k) - \frac{1}{k} < \tilde{t}_k < l^+(s_k). \tag{7.31}$$

It follows from (7.26) and (7.30) that

$$\phi(\gamma(s_k) + t_k\theta(s_k)) = \phi(\gamma(s_k)) + t_k. \tag{7.32}$$

Since $(t_k)_{k\in\mathbb{N}}$, $(\tilde{t}_k)_{k\in\mathbb{N}}$ are uniformly bounded by $2R$ and $|\theta(s_k)| = 1$ there exists a subsequence $k \to \infty$ and $\tilde{t}$, $t \in \mathbb{R}, \theta \in \mathbb{R}^2$ with $\tilde{t} \le t$, $|\theta| = 1$ such that

$$t_k \to t, \quad \tilde{t}_k \to \tilde{t}, \quad \text{and} \quad \theta(s_k) \to \theta.$$

We deduce from (7.30)-(7.32) that

$$\limsup_{k\to\infty} L^+(s_k) \le t, \tag{7.33}$$

$$\limsup_{k\to\infty} l^+(s_k) \le \tilde{t}, \tag{7.34}$$

$$\phi(\gamma(s) + t\theta) = \phi(\gamma(s)) + t. \tag{7.35}$$

Since ϕ has Lipschitz constant one it follows from (7.35) that

$$\theta \;=\; \nabla\phi(\gamma(s)) \;=\; \theta(s). \tag{7.36}$$

We therefore deduce by (7.35) that $t \leq L^+(s)$ and by (7.33) we find that

$$\limsup_{k\to\infty} L^+(s_k) \quad\leq\quad L^+(s),$$

which proves the upper-semicontinuity and therefore the measurability of L^+. By analogous arguments one proves that L^- is lower-semicontinuous and measurable. Let now $0 < \tau < \tilde{\tau}$. Since $\tilde{t}_k \to \tilde{t}$ we obtain from (7.31) that

$$\phi\big(\gamma(s_{k_i}) + \tau\theta(s_{k_i})\big) \,\in\, \mathrm{supp}(u_\varepsilon) \qquad \text{for all sufficiently large } k \in \mathbb{N}\ .$$

Since $\mathrm{supp}(u_\varepsilon)$ is closed and since ϕ is continuous this yields

$$\phi\big(\gamma(s) + \tau\theta(s)\big) \quad\in\quad \mathrm{supp}(u).$$

But $0 < \tau < \tilde{\tau}$ was arbitrary and we deduce that $\tilde{t} \leq l^+(s)$. By (7.34) this shows the upper-semicontinuity and measurability of l^+. $\square$

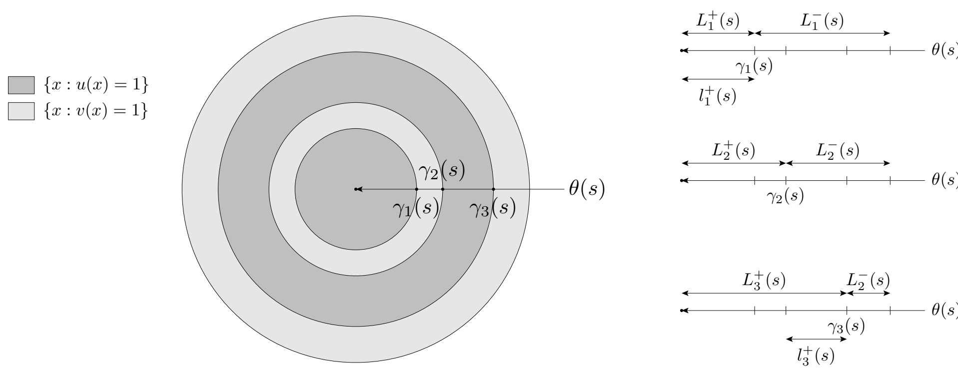

FIGURE 6. A situation where a ray crosses several times the boundary of $\mathrm{supp}(u_\varepsilon)$, given by curves $\gamma_1, \gamma_2, \gamma_3$. On the right for each of the curves γ_i the positive, negative and effective ray lengths L_i^+, L_i^-, l_i^+ is depicted. Observe that $l_2^+(s) = 0$.

The next result is similar to Lemma 16 of [16]: since rays can only intersect in a common endpoint, the ray directions vary Lipschitz continuously.

Proposition 7.6. *Let $\gamma \in \Gamma_\varepsilon$ be given as in Remark 7.2 and set for $\delta > 0$*

$$E_\delta \;:=\; \{s \in E : L^+(s), |L^-(s)| \geq \delta\}. \tag{7.37}$$

Then θ is Lipschitz continuous on E_δ with

$$\big|\theta(s_1) - \theta(s_2)\big| \quad\leq\quad \frac{2}{\delta}\,|s_1 - s_2| \qquad \textit{for all } s_1, s_2 \in E_\delta. \tag{7.38}$$

Proof. To prove (7.38) it is sufficient to consider the case $\theta(s_1) \neq \theta(s_2)$. Then there exist $t_1, t_2 \in \mathbb{R}$ such that

$$\gamma(s_1) + t_1\theta(s_1) \quad=\quad \gamma(s_2) + t_2\theta(s_2). \tag{7.39}$$

By the definition of L^+, L^- and since rays can only intersect in common endpoints we obtain

$$t_i \in \mathbb{R} \setminus \big(L^-(s_i), L^+(s_i)\big), \quad i = 1, 2. \tag{7.40}$$

By taking the determinant with $\theta(s_2)$ in (7.39) we compute

$$\det\big(\gamma(s_1) - \gamma(s_2), \theta(s_2)\big) = -t_1 \det\big(\theta(s_1), \theta(s_2)\big). \tag{7.41}$$

From (7.40) and since $s_1, s_2 \in E_\delta$ we deduce that $t_1^2 \geq \delta^2$. Taking squares in (7.41) we therefore obtain

$$\begin{aligned}(s_1 - s_2)^2 &\geq \delta^2 \det\big(\theta(s_1), \theta(s_2)\big)^2 \\ &= \delta^2 \Big[1 - (\theta(s_1) \cdot \theta(s_2))^2\Big] \\ &= \frac{\delta^2}{4} \big|\theta(s_1) - \theta(s_2)\big|^2 \big|\theta(s_1) + \theta(s_2)\big|^2. \end{aligned} \tag{7.42}$$

Next we observe that

$$\big|\theta(s_1) + \theta(s_2)\big| \geq 1 \quad \text{ if } |s_1 - s_2| \leq \delta. \tag{7.43}$$

Indeed we compute

$$\begin{aligned} 4\delta &= \phi(\gamma(s_1) + \delta\theta(s_1)) - \phi(\gamma(s_1) - \delta\theta(s_1)) + \phi(\gamma(s_2) + \delta\theta(s_2)) - \phi(\gamma(s_2) - \delta\theta(s_2)) \\ &= \phi(\gamma(s_1) + \delta\theta(s_1)) - \phi(\gamma(s_2) - \delta\theta(s_2)) + \phi(\gamma(s_2) + \delta\theta(s_2)) - \phi(\gamma(s_1) - \delta\theta(s_1)) \\ &\leq 2|\gamma(s_1) - \gamma(s_2)| + 2\delta|\theta(s_1) + \theta(s_2)| \\ &\leq 2|s_1 - s_2| + 2\delta|\theta(s_1) + \theta(s_2)| \end{aligned}$$

which proves (7.43).
By (7.42), (7.43) we deduce that (7.38) holds for $s_1, s_2 \in E_\delta$ with $|s_1 - s_2| \leq \delta$. In the case $|s_1 - s_2| > \delta$ we obtain (7.38) since $|\theta(s_1) - \theta(s_2)| \leq 2$. □

Definition 7.7. *Define two functions $\alpha, \beta : E \to \mathbb{R} \bmod 2\pi$ by requiring that*

$$\theta(s) = \begin{pmatrix} \cos\alpha(s) \\ \sin\alpha(s) \end{pmatrix}, \qquad \det\big(\gamma'(s), \theta(s)\big) = \sin\beta(s). \tag{7.44}$$

We note that

$$\sin\beta(s) \geq 0 \quad \text{ for } s \in E \text{ with } l^+(s) > 0 \tag{7.45}$$

by (7.25) and the definition of l^+.

Proposition 7.6 yields the Lipschitz continuity of θ, α on sets where the positive and negative ray lengths are strictly positive. By Kirszbraun's Theorem and Rademacher's Theorem we obtain an extension which is differentiable almost everywhere. To identify these derivatives as a property of α we prove the existence of α' in the sense of *approximate derivatives* [26, section 6.1.3].

Lemma 7.8. *The function $\alpha : E \to \mathbb{R} \bmod 2\pi$ as defined in Definition 7.7 is almost everywhere approximately differentiable and its approximate differential satisfies*

$$\sin\beta(s) - t\alpha'(s) \quad \geq \quad 0 \quad \textit{for all} \quad L^-(s) \leq t \leq L^+(s) \tag{7.46}$$

for almost all $s \in E$ with $l^+(s) > 0$.

Proof. Consider for $\delta > 0$ the set E_δ defined in (7.37). By Proposition 7.6 the restriction of θ to E_δ is Lipschitz continuous. From Kirszbraun's Theorem we deduce the existence of a Lipschitz continuous extension $\tilde{\alpha}_\delta : \mathbb{R} \to \mathbb{R} \bmod 2\pi$ of α, see for example the construction in [19]. This extension is almost everywhere differentiable by Rademacher's Theorem. By [26, Corollary 1.7.3] almost all points of E_δ have density one in E_δ and we deduce that α is approximately differentiable in these points and that the approximate differential coincides with $\tilde{\alpha}_\delta'$. Since $E = \cup_{k\in\mathbb{N}} E_{1/k}$ this proves that α is approximately differentiable almost everywhere in E.

Since a measurable function is almost everywhere approximately continuous by [26, Theorem 1.7.3], it is sufficient to prove (7.46) for $s \in E$ with $l^+(s) > 0$ such that

- α is approximately differentiable in s, and
- there exists a sequence $(s_k)_{k\in\mathbb{N}} \subset E$ with $s_k \to s$ as $k \to \infty$ and
$$L^+(s_k) \to L^+(s), \quad L^-(s_k) \to L^-(s), \quad l^+(s_k) \to l^+(s),$$
$$L^+(s_k),\, -L^-(s_k),\, l^+(s_k) > 0 \quad \text{for all } k \in \mathbb{N}\,.$$

Since (7.46) is satisfied for $\alpha'(s) = 0$ we can also assume that

- $\alpha'(s) > 0$ (the other case is analogous);
- $(\alpha(s_k) - \alpha(s))/(s_k - s) \to \alpha'(s)$, or equivalently, that $(\theta(s_k) - \theta(s))/(s_k - s) \to \theta'(s)$;
- for all $k \in \mathbb{N}$ there exist t_k, t_k^* such that
$$\gamma(s_k) + t_k\theta(s_k) \;=\; \gamma(s) + t_k^*\theta(s). \tag{7.47}$$

Analogously to (7.40), (7.41) we find from (7.47)

$$t_k \;\in\; \mathbb{R} \setminus \big(L^-(s_k), L^+(s_k)\big), \quad t_k^* \;\in\; \mathbb{R} \setminus \big(L^-(s), L^+(s)\big), \tag{7.48}$$

$$\det\big(\gamma(s_k) - \gamma(s), \theta(s)\big) \;=\; -t_k \det\big(\theta(s_k), \theta(s)\big). \tag{7.49}$$

Writing equation (7.49) as

$$\det\Big(\frac{\gamma(s_k) - \gamma(s)}{s_k - s}, \theta(s)\Big) \;=\; -t_k \det\Big(\frac{\theta(s_k) - \theta(s)}{s_k - s}, \theta(s)\Big),$$

we observe that the left-hand side converges to $\det(\gamma'(s), \theta(s)) = \sin\beta(s) \geq 0$ and the determinant on the right-hand side to $\det(\theta'(s), \theta(s)) = -\alpha'(s) < 0$. For sufficiently large k we can therefore eliminate the negative range for t_k in (7.48) and obtain

$$t_k \;\geq\; L^+(s_k).$$

Again using the negative sign of $\det\big((\theta(s_k) - \theta(s))/(s_k - s), \theta(s)\big)$ we then find

$$\det\Big(\frac{\gamma(s_k) - \gamma(s)}{s_k - s}, \theta(s)\Big) \;\geq\; -L^+(s_k) \det\Big(\frac{\theta(s_k) - \theta(s)}{s_k - s}, \theta(s)\Big)$$

Taking the limit $k \to \infty$ we deduce

$$\sin\beta(s) \;\geq\; L^+(s)\, \alpha'(s).$$

This proves the Lemma. □

For each $\gamma \in \Gamma_\varepsilon$ we define analogously to (7.29) a map ψ. Restricting these maps suitably we obtain a parametrization of $\operatorname{supp}(u_\varepsilon)$ which is essentially one-to-one.

Proposition 7.9. *Let $\Gamma_\varepsilon = \{\gamma_i : i = 1, ..., N\}$ be chosen as in Remark 7.2. For all $i = 1, ..., N$ let $L_i = L(\gamma_i)$. Then, with*

$$D_i \quad := \quad \big\{(s,t) : s \in E_i \cap [0, L_i), 0 \le t < l_i^+(s)\big\}, \tag{7.50}$$

the restrictions $\psi_i : D_i \to \mathbb{R}^2$ give, up to a Lebesgue nullset, an injective map onto $\operatorname{supp}(u_\varepsilon)$*: for almost all $x \in \operatorname{supp}(u_\varepsilon)$ there exists a unique $i \in \{1, ..., N\}$ and a unique $(s,t) \in D_i$ such that $\psi_i(s,t) = x$.*

Proof. For both the injectivity and the surjectivity it is sufficient to consider only interior points of $\operatorname{supp}(u_\varepsilon)$ that are not ray ends, since the boundary of $\operatorname{supp}(u_\varepsilon)$ and the sets of ray ends form an $\mathcal{L}^2$-nullset (Proposition 5.5).

For the surjectivity, let x be such a point in $\operatorname{supp}(u_\varepsilon)$. By (5.5) there exists $y \in \operatorname{supp}(v)$ such that

$$\phi(x) - \phi(y) \quad = \quad |x - y|$$

and it follows that x is on the interior of a ray with direction $\nabla\phi(x) = \frac{x-y}{|x-y|}$. Define

$$t \quad := \quad \min\,\{t > 0 : x - t\nabla\phi(x) \in \partial\operatorname{supp}(u_\varepsilon)\}.$$

Then there exists $i \in \{1, ..., N\}$ and $s \in E_i$ such that $l^+(s) \ge t > 0$ and

$$x \;=\; \gamma_i(s) + t\theta_i(s) \;=\; \psi_i(s,t).$$

To prove the injectivity part, let x be the same point again, and assume that there exists $j \in \{1, ..., N\}$, $(\sigma, \tau) \in D_j$ such that

$$x \;=\; \psi_i(s,t) \;=\; \psi_j(\sigma,\tau).$$

Since different rays cannot intersect in interior points such as x, the three points x, $\gamma_i(s)$, and $\gamma_j(\sigma)$ are on the same ray, and since $t > 0$ and $\tau > 0$ we have

$$\phi(\gamma_i(s)) \;<\; \phi(x) \qquad \text{and} \qquad \phi(\gamma_j(\sigma)) \;<\; \phi(x).$$

This implies that $\gamma_i(s)$ and $\gamma_j(\sigma)$ are on the same side of the ray with respect to x. On the other hand by the definition of l^+ and since $t < l^+(s)$, $\tau < l^+(\sigma)$, neither the part of the ray between x and $\gamma_i(s)$ nor the part between x and $\gamma_j(\sigma)$ contains a point of $\partial\operatorname{supp}(u_\varepsilon)$. This implies that $\gamma_i(s) = \gamma_j(\sigma)$ which proves the injectivity part of the proposition. □

Using Proposition 7.6 and Proposition 7.9 we can justify the following transformation formula for integrals.

Lemma 7.10. *Let $\Gamma_\varepsilon = \{\gamma_i\}_{i=1,\dots,N}$ be as in Remark 7.2 and D_i as defined in (7.50). Then for all $g \in L^1(\mathbb{R}^2)$*

$$\int g(x)u_\varepsilon(x)\,dx = \sum_{i=1}^{N} \int_{D_i} g(\psi_i(s,t))\frac{1}{\varepsilon}\big(\sin\beta_i(s) - t\alpha_i'(s)\big)\,dt\,ds \tag{7.51}$$

holds.

Proof. We deduce from Proposition 7.6 that for all $i \in \{1, ..., N\}$ the functions θ_i and ψ_i are approximately differentiable on D_i with approximate differentials

$$\theta_i'(s) \;=\; \alpha_i'(s)\begin{pmatrix} -\sin\alpha_i(s) \\ \cos\alpha_i(s) \end{pmatrix},$$

$$\partial_s\psi_i(s,t) \;=\; \gamma_i'(s) + t\theta_i'(s), \qquad \partial_t\psi_i(s,t) \;=\; \theta_i'(s).$$

This yields that

$$\begin{aligned}|\det D\psi_i(s,t)| &= \big|\det\big(\gamma_i'(s),\theta_i(s)\big) + t\det\big(\theta_i'(s),\theta_i(s)\big)\big| \\ &= |\sin\beta_i(s) - t\alpha_i'(s)| \\ &= \sin\beta_i(s) - t\alpha_i'(s)\end{aligned}$$

where we have used (7.46) in the last equality.
We then deduce from the generalized transformation formula [8, Remark 5.5.2] that

$$\int_{D_i} g(\psi_i(s,t))\frac{1}{\varepsilon}\big(\sin\beta_i(s) - t\alpha_i'(s)\big)\,dt\,ds \;=\; \int_{\psi_i(D_i)} g(x)u_\varepsilon(x)\,dx.$$

Summing these equalities over $i = 1, ..., N$ we deduce by Proposition 7.9 that (7.51) holds. □

Often it is more convenient to work not in *length coordinates* but rather in *mass coordinates* which are defined as follows.

Definition 7.11. *For $\gamma \in \Gamma_\varepsilon$ and $s \in \mathbb{R}$ we define a map $\mathfrak{m}_s : \mathbb{R} \to \mathbb{R}$ and a map $M : \mathbb{R} \to \mathbb{R}$ by*

$$\mathfrak{m}(s,t) := \begin{cases} \frac{t}{\varepsilon}\sin\beta(s) - \frac{t^2}{2\varepsilon}\alpha'(s) & \text{if } l^+(s) > 0, \\ 0 & \text{otherwise.}\end{cases} \tag{7.52}$$

$$M(s) := \mathfrak{m}(s, l^+(s)). \tag{7.53}$$

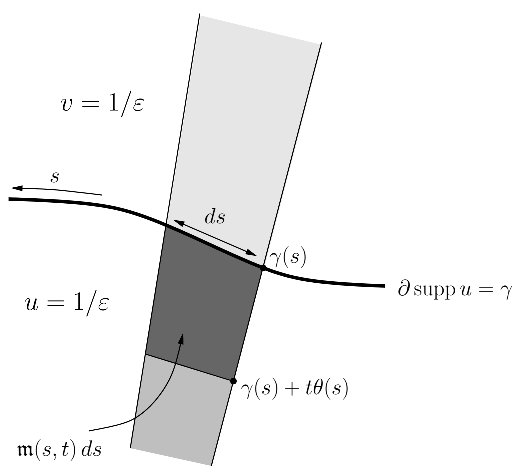

FIGURE 7. The function $\mathfrak{m}$. Two rays delimit a section of u-mass (below the curve γ, in this figure) that is transported to the associated section of v-mass. For given $t > 0$, $\mathfrak{m}(s,t)\,ds$ is the amount of u-mass contained between the two rays and stretching from $\gamma(s)$ to the point $\gamma(s) + t\theta(s)$. For $t < 0$, $\mathfrak{m}(t,s)$ is the corresponding amount of v-mass, with a minus sign.

Lemma 7.12. *The map $\mathfrak{m}(s,\cdot)$ is strictly monotone on $\big(L^-(s), L^+(s)\big)$ with inverse*

$$\mathfrak{t}(s,m) = \frac{\sin\beta(s)}{\alpha'(s)}\Big[1 - \Big(1 - \frac{2\alpha'(s)\varepsilon}{\sin^2\beta(s)}m\Big)^{\frac{1}{2}}\Big]. \tag{7.54}$$

Let $\Gamma_\varepsilon = \{\gamma_i\}_{i=1,\dots,N}$ be given as in Remark 7.2, set $L_i = L(\gamma_i)$, and let M_i be the mass function (7.53) for curve γ_i. We obtain for $g \in L^1(\mathbb{R}^2)$ that

$$\int g(x)u_\varepsilon(x)\,dx = \sum_{i=1}^N \int_0^{L_i} \int_0^{M_i(s)} g(\psi_i(s, \mathfrak{t}_i(s,m)))\,dm\,ds. \tag{7.55}$$

In particular, the total mass of u_ε is given by

$$\int u_\varepsilon(x)\,dx \quad = \quad \sum_{i=1}^N \int_0^{L_i} M_i(s)\,ds. \tag{7.56}$$

Proof. The calculation of the inverse is straightforward. For the transformation formula (7.55) we combine (7.51) with the remark that

$$\frac{\partial}{\partial t}\,\mathfrak{m}(s,t) = \frac{1}{\varepsilon}\big(\sin\beta_i(s) - t\alpha_i'(s)\big).$$

Note that the integration interval $0 < t < l_i^+(s)$ in the definition (7.50) of D_i transforms to $0 < m < M_i(s)$. □

Typically, the optimal transport map is built by gluing solutions of one-dimensional transport problems on single rays together. We observe a similar structure here.

Proposition 7.13. *Let $\gamma \in \Gamma_\varepsilon$ be as in Remark 7.2 and S be the optimal transport map from u_ε to v_ε, as in Proposition 5.3. Define for $s \in E$ with $l^+(s) > 0$ an interval $I(s) \subset \mathbb{R}$ and measures f_s^+, f_s^- on $I(s)$,*

$$\begin{aligned} &I(s) := \Big(\mathfrak{m}_s\big(L^-(s)\big), \mathfrak{m}_s\big(L^+(s)\big)\Big),\\ &df_s^+ := u_\varepsilon\big(\psi(s,\mathfrak{t}(s,m))\big)\,dm, \qquad df_s^- := v_\varepsilon\big(\psi(s,\mathfrak{t}(s,m))\big)\,dm. \end{aligned}$$

Moreover, define a map

$$\hat{S} : I(s) \cap \operatorname{supp}(f_s^+) \quad \to \quad I(s) \cap \operatorname{supp}(f_s^-),$$

by the equation

$$S\big(\psi(s,\mathfrak{t}(s,m))\big) \;=\; \psi\big(s,\mathfrak{t}(s,\hat{S}(m))\big) \qquad \text{for } m \in I \cap \operatorname{supp}(f_s^+). \tag{7.57}$$

Then for almost all $s \in E$ with $l^+(s) > 0$ the map $\hat{S}$ is the unique monotone transport map pushing f_s^+ forward to f_s^-.

Proof. It is sufficient to prove the claim for almost all $s \in \{l^+ > k^{-1}\}$, $k \in \mathbb{N}$ arbitrary. Let $A \subset E$ be any set such that

$$A \;\subset\; \{s \in E : l^+(s) > k^{-1}\}, \quad \operatorname{diam}(A) \;<\; k^{-1} \tag{7.58}$$

We define rays and a set of rays

$$\begin{aligned} \mathcal{R}(s) \;&:=\; \{\gamma(s) + \tau\theta(s) : \tau \in (L^-(s), L^+(s))\},\\ \mathcal{R}(A) \;&:=\; \bigcup_{s\in A} \mathcal{R}(s). \end{aligned}$$

Since different rays can only intersect in a common endpoint, two rays $\mathcal{R}(s), \mathcal{R}(\sigma)$ with $s,\sigma \in A$, $s \neq \sigma$ can only be disjoint or identical. The latter case is excluded by (7.58): since $l^+(s), l^+(\sigma) > k^{-1}$ the distance between $\gamma(s)$ and $\gamma(\sigma)$ has to be at least k^{-1} in contradiction to the assumption on the diameter of A in (7.58). Therefore the union $\{\mathcal{R}(s) : s \in A\}$ is pairwise disjoint.

Let now $(g_j)_{j\in\mathbb{N}}$ be a dense subset of $C^0(\mathbb{R})$. Since S pushes $u_\varepsilon|_{\mathcal{R}(A)}$ forward to $v_\varepsilon|_{\mathcal{R}(A)}$ we obtain for all $j \in \mathbb{N}$

$$\int_{\mathcal{R}(A)} g_j(\phi(S(x)))u_\varepsilon(x)\,dx \quad = \quad \int_{\mathcal{R}(A)} g_j(\phi(x))v_\varepsilon(x)\,dx. \tag{7.59}$$

Repeating the changes of variables of Lemmas 7.10 and 7.12, and using the fact that $\{\mathcal{R}(s) : s \in A\}$ is pairwise disjoint we obtain

$$\begin{aligned}\int_A \int_{I(s)} & g_j\Big(\phi\big(S(\psi(s,\mathfrak{t}(s,m)))\big)\Big)u_\varepsilon\big(\psi(s,\mathfrak{t}(s,m))\big)\,dm\,ds \\ &= \int_A \int_{I(s)} g_j\Big(\phi\big(\psi(s,\mathfrak{t}(s,m))\big)\Big)v_\varepsilon\big(\psi(s,\mathfrak{t}(s,m))\big)\,dm\,ds,\end{aligned}$$

and by using the definition of $\hat{S}$ (7.57) and the linearity of ϕ along rays we write this as

$$\begin{aligned}\int_A \int_{I(s)} & g_j\big(\phi(\gamma(s)) + \mathfrak{t}(s,\hat{S}(m))\big)u_\varepsilon\big(\psi(s,\mathfrak{t}(s,m))\big)\,dm\,ds \\ &= \int_A \int_{I(s)} g_j\big(\phi(\gamma(s)) + \mathfrak{t}(s,m)\big)v_\varepsilon\big(\psi(s,\mathfrak{t}(s,m))\big)\,dm\,ds,\end{aligned}$$

Since A as above was arbitrary we deduce that for almost all $s \in \{l^+ > 0\}$ and all $j \in \mathbb{N}$

$$\int_{I(s)} g_j\big(\phi(\gamma(s)) + \mathfrak{t}(s,\hat{S}(m))\big)\,df_s^+(m) \quad = \quad \int_{I(s)} g_j\big(\phi(\gamma(s)) + \mathfrak{t}(s,m)\big)\,df_s^-(m).$$

Since $(g_j)_{j\in\mathbb{N}}$ is dense in $C^0(\mathbb{R})$ and since for fixed s the function $m \mapsto \phi(\gamma(s)) + \mathfrak{t}(s,m)$ is a homeomorphism it follows that

$$\int_{I(s)} g(\hat{S}(m))\,df_s^+(m) \quad = \quad \int_{I(s)} g(m)\,df_s^-(m)$$

for all $g \in C^0(\mathbb{R})$. Therefore $\hat{S}$ pushes f_s^+ forward to f_s^-. The monotonicity of $\hat{S}$ follows from the monotonicity of S. By [5, Theorem 3.1] the map $\hat{S}$ is the unique monotone transport map from f_s^+ to f_s^-. $\square$

Lemma 7.14. *Let $\gamma \in \Gamma_\varepsilon$ as in Remark 7.2. Then for almost all $s \in E$ with $l^+(s) > 0$ we obtain that*

$$\big|\psi\big(s,\mathfrak{t}(s,m)\big) - S\big(\psi\big(s,\mathfrak{t}(s,m)\big)\big)\big| \;\geq\; \mathfrak{t}(s,m) - \mathfrak{t}(s,m-M(s)) \tag{7.60}$$

for all $0 < m < M(s)$.

Proof. We let $\hat{S}$ be as in (7.57). Using the monotonicity of $\mathfrak{t}(s,.)$ we obtain

$$\begin{aligned}\big|\psi\big(s,\mathfrak{t}(s,m)\big) - S\big(\psi\big(s,\mathfrak{t}(s,m)\big)\big)\big| &= \big|\psi\big(s,\mathfrak{t}(s,m)\big) - \psi\big(s,\mathfrak{t}(s,\hat{S}(m))\big)\big| \\ &= \mathfrak{t}(s,m) - \mathfrak{t}(s,\hat{S}(m)).\end{aligned} \tag{7.61}$$

By Proposition 7.13 the map $\hat{S}$ is the unique monotone transport map from f_s^+ to f_s^-. In particular

$$f_s^-\big(\big[\hat{S}(m),\hat{S}(M(s))\big]\big) \;=\; f_s^+\big([m,M(s)]\big) \tag{7.62}$$

holds. Since f_s^+ restricted to the set $[0, M(s)]$ coincides with the Lebesgue measure and since $0 \leq f_s^-, f_s^+ \leq \mathcal{L}^1$ we deduce from (7.62) that

$$\hat{S}(M(s)) - \hat{S}(m) \geq M(s) - m \tag{7.63}$$

for any $0 \leq m \leq M(s)$. The value $\hat{S}(M(s))$ is less or equal than zero since $\hat{S}$ is decreasing and maps onto the support of f_s^-. This implies by (7.63) that $\hat{S}(m) \leq m - M(s)$. By the montonicity of $\mathfrak{t}_s$ we deduce

$$\mathfrak{t}(s,m) - \mathfrak{t}(s, \hat{S}(m)) \geq \mathfrak{t}(s,m) - \mathfrak{t}(s, m - M(s)).$$

Together with (7.61) this proves (7.60). □

By Lemma 7.6 the ray directions vary Lipschitz continuously on sets where the positive and negative ray lengths are bounded from below. We obtain a Lipschitz bound also on sets where $M(\cdot)$ is bounded from below.

Lemma 7.15. *Let $\gamma \in \Gamma_\varepsilon$ be as in Remark 7.2. Then the inequality*

$$\sin\beta(s) \min\big(L^+(s), |L^-(s)|\big) \quad \geq \quad \frac{\varepsilon}{2} M(s) \tag{7.64}$$

holds. In particular, for all $0 < \kappa < 1$ the function α is Lipschitz continuous on the set $\{s : M(s) \geq (1-\kappa)\}$ and

$$|\alpha'(s)| \quad \leq \quad \frac{2}{\varepsilon(1-\kappa)} \tag{7.65}$$

holds for almost all $s \in \mathbb{R}$ with $M(s) \geq (1-\kappa)$.

Proof. By (7.46) we get that $\sin\beta(s) \geq 0$ for $s \in E$ with $l^+(s) > 0$ and that $\alpha'(s) = 0$ if $\sin\beta(s) = 0$. Since in the latter case (7.64) holds it is sufficient to assume that $\sin\beta(s) > 0$. We consider first the case $\alpha'(s) \leq 0$. From (7.52) and (7.53) we obtain

$$\begin{aligned} M(s) \quad &= \quad \frac{1}{\varepsilon}\Big(l^+(s)\sin\beta(s) - \frac{1}{2} l^+(s)^2 \alpha'(s)\Big) \\ &\leq \quad \frac{\sin\beta(s)}{\varepsilon} l^+(s) - \frac{\sin\beta(s)}{\varepsilon} \cdot \frac{l^+(s)^2}{2L^-(s)}. \end{aligned} \tag{7.66}$$

By Proposition 7.13 the estimate

$$M(s) = f_s^+((0, M(s)) \leq f_s^-\big(I(s) \cap (L^-(s), 0]\big)$$

holds and we deduce that

$$\begin{aligned} M(s) \quad &\leq \quad \int_{L^-(s)}^0 \frac{1}{\varepsilon}\big(\sin\beta(s) - t\alpha'(s)\big)\, dt \\ &\leq \quad \frac{|L^-(s)|}{\varepsilon} \sin\beta(s) \\ &= \quad -\frac{\sin\beta(s)}{\varepsilon} L^-(s). \end{aligned} \tag{7.67}$$

Together with (7.66) we deduce

$$M(s) \quad \leq \quad \frac{\sin\beta(s)}{\varepsilon} l^+(s) + \frac{1}{2M(s)}\Big(\frac{\sin\beta(s)}{\varepsilon} l^+(s)\Big)^2.$$

Therefore $\xi = \frac{\sin\beta(s)}{\varepsilon} l^+(s)$ satisfies the inequality

$$\xi^2 + 2M(s)\xi - 2M(s)^2 \quad \geq \quad 0$$

and we obtain that

$$\frac{\sin\beta(s)}{\varepsilon} l^+(s) \quad \geq \quad (\sqrt{3}-1)M(s). \tag{7.68}$$

Together with (7.67) and $l^+(s) \leq L^+(s)$ this proves (7.64) in the case that $\alpha'(s) \leq 0$. Next we assume that $\alpha'(s) > 0$. From (7.52), (7.53) we observe that

$$M(s) \quad \leq \quad \frac{1}{\varepsilon} L^+(s) \sin\beta(s). \tag{7.69}$$

To prove (7.64) also for $L^-(s)$ let us assume—without loss of generality—that $|L^-(s)| \leq \frac{\varepsilon}{\sin\beta(s)} M(s)$. We then deduce that

$$|L^-(s)| \leq L^+(s) \tag{7.70}$$

and compute

$$\begin{aligned} M(s) \quad &\leq \quad \int_{L^-(s)}^0 v_\varepsilon(\psi(s,t))\big(\sin\beta(s) - t\alpha'(s)\big)\,dt \\ &\leq \quad \int_{L^-(s)}^0 \frac{1}{\varepsilon}\big(\sin\beta(s) - L^-(s)\alpha'(s)\big)\,dt \;=\; \frac{|L^-(s)|}{\varepsilon}\big(\sin\beta(s) - L^-(s)\alpha'(s)\big). \end{aligned}$$

By (7.70) and (7.46) we deduce that

$$M(s) \leq \frac{|L^-(s)|}{\varepsilon}\big(\sin\beta(s) + L^+(s)\alpha'(s)\big) \leq 2|L^-(s)|\frac{\sin\beta(s)}{\varepsilon}.$$

Together with (7.69) this proves (7.64) in the case $\alpha'(s) > 0$.

The estimate (7.65) follows from (7.64) and (7.46). □

7.2. **Estimate from below.** Let S be the optimal transport map and $\phi \in \mathrm{Lip}_1(\mathbb{R}^2)$ be an optimal Kantorovich potential for the mass transport from u_ε to v_ε as in Proposition 5.3.

Using Lemma 7.12 and Lemma 7.14 we obtain for the Monge-Kantorovich distance between u to v that

$$\begin{aligned} d_1(u,v) \quad &= \quad \int_{\mathbb{R}^2} |x - S(x)| u_\varepsilon(x)\,dx \\ &= \quad \sum_{i=1}^N \int_0^{L_i} \int_0^{M_i(s)} \big|\psi_i(s,\mathfrak{t}_i(s,m)) - S\big(\psi_i(s,\mathfrak{t}_i(s,m))\big|\,dm\,ds \\ &\geq \quad \sum_{i=1}^N \int_0^{L_i} \int_0^{M_i(s)} \big[\mathfrak{t}_i(s,m) - \mathfrak{t}_i(s,m-M_i(s))\big]\,dm. \end{aligned} \tag{7.71}$$

We further estimate the right-hand side of this inequality.

Lemma 7.16. *Let Γ_ε be a disjoint system of curves parametrized by arclength as in Remark 7.2 and let $\gamma \in \Gamma_\varepsilon$. Then we obtain for all $s \in E$ with $l^+(s) > 0$ and $\sin\beta(s) > 0$ that*

$$\begin{aligned} &\int_0^{M(s)} \big[\mathfrak{t}(s,m) - \mathfrak{t}(s,m-M(s))\big]\,dm \\ &= \frac{\varepsilon}{\sin\beta(s)} M(s)^2 + \frac{\varepsilon^3}{4\sin^5\beta(s)} \alpha'(s)^2 M(s)^4 + R(s)\varepsilon^5, \end{aligned} \tag{7.72}$$

where

$$0 \le R(s) \le \frac{7}{9}\frac{M(s)^6}{\sin\beta(s)^9}\alpha'(s)^4. \tag{7.73}$$

Proof. We compute for $r \in \mathbb{R}$

$$\begin{aligned}\int_0^r \mathfrak{t}(s,m)\,dm &= \int_0^r \frac{\sin\beta(s)}{\alpha'(s)}\Big[1-\Big(1-\frac{2\alpha'(s)\varepsilon}{\sin^2\beta(s)}m\Big)^{\frac12}\Big]\,dm\\ &= \frac{\sin^3\beta(s)}{3\alpha'(s)^2\varepsilon}\Big[\Big(1-\frac{2\alpha'(s)\varepsilon}{\sin^2\beta(s)}r\Big)^{\frac32}+\frac{3\varepsilon\alpha'(s)}{\sin^2\beta(s)}r-1\Big] \end{aligned}\tag{7.74}$$

and thus

$$\begin{aligned}&\int_0^{M(s)}\big[\mathfrak{t}(s,m)-\mathfrak{t}(s,m-M(s))\big]\,dm\\ &=\int_0^{M(s)}\mathfrak{t}(s,m)\,dm+\int_0^{-M(s)}\mathfrak{t}(s,m)\,dm\\ &=\frac{\sin^3\beta(s)}{3\alpha'(s)^2\varepsilon}\left[\Big(1+\frac{2\alpha'(s)\varepsilon}{\sin^2\beta(s)}M(s)\Big)^{\frac32}+\Big(1-\frac{2\alpha'(s)\varepsilon}{\sin^2\beta(s)}M(s)\Big)^{\frac32}-2\right]\end{aligned}\tag{7.75}$$

By a Taylor expansion we observe that for all $\xi > 0$

$$(1+\xi)^{\frac32}+(1-\xi)^{\frac32}-2 \;=\; \frac34\xi^2+\frac{3}{64}\xi^4+\frac79 2^{-6}\zeta(\xi)^6 \tag{7.76}$$

with $0 \le \zeta(\xi) \le \xi$. Using this expansion in (7.75) we obtain (7.72), (7.73). □

Putting all information together we derive the following estimate.

Proposition 7.17. *Let $\partial\,\mathrm{supp}(u_\varepsilon)$ be given by a disjoint system Γ_ε of closed curves parametrized by arclength as in Remark 7.2. Then the lower bound*

$$\begin{aligned}\mathcal{G}_\varepsilon(u_\varepsilon,v_\varepsilon) \;&\ge\; \sum_{i=1}^N\int_0^{L_i}\left[\frac{1}{\varepsilon^2}\Big(\frac{1}{\sin\beta_i(s)}-1\Big)M_i(s)^2+\frac{1}{\varepsilon^2}\big(M_i(s)-1\big)^2\right]ds\\ &\quad+\sum_{i=1}^N\int_0^{L_i}\frac{1}{4\sin\beta_i(s)}\Big(\frac{M_i(s)}{\sin\beta_i(s)}\Big)^4\alpha_i'(s)^2\,ds\end{aligned}\tag{7.77}$$

holds.

Proof. Since $L_i = L(\gamma_i)$ and by (7.56) we obtain that

$$\varepsilon\int_{\mathbb{R}^2}|\nabla u_\varepsilon| = \sum_{i=1}^N L_i \qquad\text{and}\qquad 1 = \int_{\mathbb{R}^2}u_\varepsilon = \sum_{i=1}^N\int_0^{L_i}M_i(s)\,ds.$$

From (7.71) and Lemma 7.16 we therefore obtain

$$\begin{aligned}\mathcal{G}_\varepsilon(u_\varepsilon,v_\varepsilon) \;&\ge \sum_{i=1}^N\int_0^{L_i}\Big(\frac{1}{\varepsilon^2\sin\beta_i(s)}M_i(s)^2-\frac{2}{\varepsilon^2}M_i(s)+\frac{1}{\varepsilon^2}\Big)\\ &\quad+\sum_{i=1}^N\int_0^{L_i}\frac{1}{4\sin^5\beta_i(s)}\alpha_i'(s)^2M_i(s)^4,\end{aligned}$$

and observing that

$$\frac{1}{\sin\beta_i(s)}M_i(s)^2 - 2M_i(s) + 1 \;=\; \Big(\frac{1}{\sin\beta_i(s)} - 1\Big)M_i(s)^2 + \big(M_i(s) - 1\big)^2$$

the estimate (7.77) follows. □

7.3. Compactness of the boundary curves and proof of the lim-inf estimate. In this subsection we prove that the boundaries of $\operatorname{supp}(u_\varepsilon)$ converge to a system of closed curves which satisfies the lim-inf estimate. We later identify this limit with the limit of $u_\varepsilon\mathcal{L}^2$.

Let $\partial\operatorname{supp}(u_\varepsilon)$ be given by a system of closed curves $\Gamma_\varepsilon = \{\gamma_{\varepsilon,i}\}_{i=1,\dots,N(\varepsilon)}$ parametrized by arclength as in Remark 7.2. In particular we recall that $L_{\varepsilon,i} = L(\gamma_{\varepsilon,i})$.

The total length of the boundary of $\operatorname{supp}(u_\varepsilon)$ is given by $|\Gamma_\varepsilon|$ and we obtain from (7.2) and (7.77)

$$|\Gamma_\varepsilon| \;=\; \sum_{i=1}^{N(\varepsilon)} L_{\varepsilon,i} \;=\; \varepsilon\int_{\mathbb{R}^2}|\nabla u_\varepsilon| \;\leq\; 2 + \varepsilon^2\mathcal{G}_\varepsilon(u_\varepsilon,v_\varepsilon) \;\leq\; 2 + \varepsilon^2\Lambda. \tag{7.78}$$

This implies the convergence of the measures $\varepsilon|\nabla u_\varepsilon| = \mu_{\Gamma_\varepsilon}$ as Radon measures on $\mathbb{R}^2$ for a subsequence $\varepsilon \to 0$.

We first define modified curves $\tilde\gamma_{\varepsilon,i}$ which are uniformly bounded in $W^{2,2}\big(0,L_{\varepsilon,i}\big)$.

Definition 7.18. *Let $\gamma \in \Gamma_\varepsilon$ and $0<\lambda<1$. Choose a periodic Lipschitz continuous function $\tilde\alpha_{\varepsilon,i} : \mathbb{R} \to \mathbb{R} \bmod 2\pi$ with period $L_{\varepsilon,i}$ such that*

$$\tilde\alpha_{\varepsilon,i} \;=\; \alpha_{\varepsilon,i} \quad \text{on} \quad \{s\in\mathbb{R} : M_{\varepsilon,i}(s) \geq 1-\lambda\}, \tag{7.79}$$

$$|\tilde\alpha'_{\varepsilon,i}| \;\leq\; \frac{2}{\varepsilon(1-\lambda)} \tag{7.80}$$

and set

$$\tilde\theta_{\varepsilon,i} \;=\; \begin{pmatrix}\cos\alpha_{\varepsilon,i}\\ \sin\alpha_{\varepsilon,i}\end{pmatrix}, \qquad \tilde\theta^\perp_{\varepsilon,i} \;=\; \begin{pmatrix}\sin\alpha_{\varepsilon,i}\\ -\cos\alpha_{\varepsilon,i}\end{pmatrix}.$$

We then define $\tilde\gamma_{\varepsilon,i} : \mathbb{R}\to\mathbb{R}^2$ to be the curve which satisfies

$$\tilde\gamma_{\varepsilon,i}(0) \;=\; \gamma_{\varepsilon,i}(0), \tag{7.81}$$

$$\tilde\gamma'_{\varepsilon,i}(s) \;=\; \tilde\theta^\perp_{\varepsilon,i}(s) \quad \text{for all } s\in\mathbb{R}. \tag{7.82}$$

Remark 7.19. By Proposition 7.6 the function $\alpha_{\varepsilon,i}$ is Lipschitz continuous on the set $\{M_{\varepsilon,i} > 1-\lambda\}$. By Kirszbraun's Theorem and (7.65) an extension $\tilde\alpha_{\varepsilon,i}$ as in Definition 7.18 exists.

The curves $\tilde\gamma_{\varepsilon,i}$ are not necessarily closed (or, equivalently, the functions $\tilde\gamma_{\varepsilon,i}$ are not necessarily periodic); the restriction $\tilde\gamma_{\varepsilon,i}|_{[0,L_{\varepsilon,i})}$ may also have self-intersections. However the *tangents* of these curves, the functions $\tilde\gamma'_{\varepsilon,i}$, are periodic with period $L_{\varepsilon,i}$.

Lemma 7.20. *Let $\varepsilon>0$, $0<\lambda<1$, $1\leq p\leq 2$, and consider for $i\in\{1,\dots,N(\varepsilon)\}$ functions $\tilde\alpha_{\varepsilon,i}$ and modified curves $\tilde\gamma_{\varepsilon,i}$ as in Definition 7.18. Then there exists a*

constant $C(\lambda,\Lambda)$, which is independent of ε, such that

$$\sum_{i=1}^{N(\varepsilon)} \int_0^{L_{\varepsilon,i}} |\gamma'_{\varepsilon,i}(s) - \tilde{\gamma}'_{\varepsilon,i}(s)|\, ds \leq \varepsilon C(\lambda,\Lambda) \tag{7.83}$$

$$\sum_{i=1}^{N(\varepsilon)} \int_0^{L_{\varepsilon,i}} |\tilde{\gamma}''_{\varepsilon,i}(s)|^2\, ds \leq C(\lambda,\Lambda), \tag{7.84}$$

$$\begin{aligned}\sum_{i=1}^{N(\varepsilon)} \int_0^{L_{\varepsilon,i}} |\tilde{\gamma}''_{\varepsilon,i}(s)|^p\, ds \leq{}& \Big[\sum_{i=1}^{N(\varepsilon)} L_{\varepsilon,i}\Big]^{1-\frac{p}{2}} \Big[4(1-\lambda)^{-4}\mathcal{G}_\varepsilon(u_\varepsilon,v_\varepsilon)\Big]^{\frac{p}{2}} \\ &+ 4\lambda^{-2}(1-\lambda)^{-2}\varepsilon^{2-p}\mathcal{G}_\varepsilon(u_\varepsilon,v_\varepsilon).\end{aligned} \tag{7.85}$$

Proof. Dropping the indexes ε, i for a moment and letting $\theta(s)^\perp = (\sin\alpha(s), -\cos\alpha(s))^T$ we compute that

$$\begin{aligned}\int_0^L |\gamma'(s) - \tilde{\gamma}'(s)|\, ds \quad\leq\quad & \int_0^L \mathcal{X}_{\{M\geq 1-\lambda\}}(s)\Big|\gamma'(s) - \theta(s)^\perp\Big|\, ds \\ & +2\int_0^L \mathcal{X}_{\{M<1-\lambda\}}(s)\, ds\end{aligned} \tag{7.86}$$

and

$$\begin{aligned}|\gamma'(s) - \theta(s)^\perp|^2 \;=\; & 2 - 2\gamma'(s)\cdot\theta(s)^\perp \\ = 2 - 2\det\big(\gamma'(s),\theta(s)\big) \;=\; & 2\sin\beta(s)\Big(\frac{1}{\sin\beta(s)} - 1\Big),\end{aligned}$$

which implies

$$\int_0^L \mathcal{X}_{\{M\geq 1-\lambda\}}(s)\Big|\gamma'(s) - \theta(s)^\perp\Big|^2\, ds \leq 2(1-\lambda)^{-2}\int_0^L \Big(\frac{1}{\sin\beta(s)} - 1\Big)M(s)^2\, ds. \tag{7.87}$$

Moreover we calculate

$$\int_0^L \mathcal{X}_{\{M<1-\lambda\}}(s)\, ds \quad\leq\quad \varepsilon^2\lambda^{-2}\int_0^L \frac{1}{\varepsilon^2}\big(1 - M(s)\big)^2\, ds. \tag{7.88}$$

By (7.87), (7.88) we obtain from (7.86) that

$$\begin{aligned}\int_0^L |\gamma'(s) - \tilde{\gamma}'(s)|\, ds \quad\leq\quad & \sqrt{2L}\varepsilon(1-\lambda)^{-1}\Big(\int_0^L \frac{1}{\varepsilon^2}\Big(\frac{1}{\sin\beta(s)} - 1\Big)M(s)^2\, ds\Big)^{1/2} \\ & +\varepsilon^2\lambda^{-2}\int_0^L \frac{1}{\varepsilon^2}\big(1 - M(s)\big)^2\, ds.\end{aligned}$$

Summing the corresponding inequalities for $i = 1, ..., N(\varepsilon)$, and using Hölder's inequality, we deduce that

$$\begin{aligned}
&\sum_{i=1}^{N(\varepsilon)} \int_0^{L_{\varepsilon,i}} |\gamma'_{\varepsilon,i}(s) - \tilde{\gamma}'_{\varepsilon,i}(s)|\,ds \\
&\quad\leq \sqrt{2}\varepsilon(1-\lambda)^{-1}\Big(\sum_{i=1}^{N(\varepsilon)} L_i\Big)^{1/2}\Big(\sum_{i=1}^{N(\varepsilon)}\int_0^{L_{\varepsilon,i}} \frac{1}{\varepsilon^2}\Big(\frac{1}{\sin\beta_{\varepsilon,i}(s)} - 1\Big) M_{\varepsilon,i}(s)^2\,ds\Big)^{1/2} \\
&\qquad\quad + \varepsilon^2\lambda^{-2}\sum_{i=1}^{N(\varepsilon)}\int_0^{L_{\varepsilon,i}} \frac{1}{\varepsilon^2}\big(1 - M_{\varepsilon,i}(s)\big)^2\,ds. \qquad (7.89) \\
&\quad\leq \sqrt{2}\varepsilon(1-\lambda)^{-1}\Big(\sum_{i=1}^{N(\varepsilon)} L_i\Big)^{1/2}\Lambda^{\frac{1}{2}} + \varepsilon^2\lambda^{-2}\Lambda, \qquad (7.90)
\end{aligned}$$

where we have used (7.2) and (7.77). By (7.78) the inequality (7.83) follows from (7.90).
Dropping indexes ε, i again we obtain that

$$\begin{aligned}
\int_0^L |\tilde{\gamma}''(s)|^p\,ds \;&=\; \int_0^L \mathcal{X}_{\{M\geq 1-\lambda\}}(s)|\alpha'(s)|^p\,ds \\
&\qquad + \int_0^L \mathcal{X}_{\{M<1-\lambda\}}(s)|\tilde{\alpha}'(s)|^p\,ds \\
&\leq\; L^{1-\frac{p}{2}}\Big[4(1-\lambda)^{-4}\int_0^L \frac{1}{4\sin\beta(s)}\Big(\frac{M(s)}{\sin\beta(s)}\Big)^4|\alpha'(s)|^2\,ds\Big]^{\frac{p}{2}} \\
&\qquad + 4(1-\lambda)^{-2}\frac{1}{\varepsilon^p}\int_0^L \mathcal{X}_{\{M<1-\lambda\}}(s)\,ds,
\end{aligned}$$

where we have used (7.80) in the last inequality. We can further estimate the second term on the right-hand side by (7.88),

$$4(1-\lambda)^{-2}\frac{1}{\varepsilon^p}\int_0^L \mathcal{X}_{\{M<1-\lambda\}}(s)\,ds \;\leq\; 4\lambda^{-2}(1-\lambda)^{-2}\varepsilon^{2-p}\int_0^L \frac{1}{\varepsilon^2}(1-M(s))^2\,ds.$$

Putting everything together and summing over $i = 1, ..., N(\varepsilon)$ we obtain

$$\begin{aligned}
\sum_{i=1}^{N(\varepsilon)}\int_0^{L_{\varepsilon,i}} |\tilde{\gamma}''_{\varepsilon,i}(s)|^p\,ds \;&\leq\; \Big[\sum_{i=1}^{N(\varepsilon)} L_{\varepsilon,i}\Big]^{1-\frac{p}{2}}\Big[4(1-\lambda)^{-4}\mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon)\Big]^{\frac{p}{2}} \\
&\qquad + 4\lambda^{-2}(1-\lambda)^{-2}\varepsilon^{2-p}\mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon)
\end{aligned}$$

which is (7.85). Setting $p = 2$ and using (7.2) the estimate (7.84) follows. □

The next result shows that those curves $\gamma_{\varepsilon,i}$ that are 'too short' do not contribute to the limit.

Lemma 7.21. *There exists $\delta = \delta(\Lambda)$ and $C = C(\Lambda)$ independent of ε such that for the index set*

$$J(\varepsilon) \;:=\; \big\{i \in \{1, ..., N(\varepsilon)\} : L_{\varepsilon,i} \leq \delta\big\}$$

the estimate

$$\sum_{i\in J(\varepsilon)} L_{i,\varepsilon} \;\leq\; \varepsilon C(\Lambda) \qquad (7.91)$$

holds.

Proof. Let $\varepsilon > 0$ and $i \in J(\varepsilon)$ be fixed and consider $\tilde{\alpha}_{\varepsilon,i}$ and $\tilde{\gamma}_{\varepsilon,i}$ as in Definition 7.18 with $\lambda = 1/2$. We drop for a moment the indexes ε, i. We then have

$$\begin{aligned} L &= \int_0^L |\tilde{\gamma}'(s)|^2 \, ds \\ &= \int_0^L \tilde{\gamma}'(s) \cdot \big(\tilde{\gamma}(s) - \tilde{\gamma}(0)\big)' \, ds \\ &= -\int_0^L \tilde{\gamma}''(s)\big(\tilde{\gamma}(s) - \tilde{\gamma}(0)\big) \, ds + \tilde{\gamma}'(L) \cdot \big(\tilde{\gamma}(L) - \tilde{\gamma}(0)\big) \\ &\leq \int_0^L s|\tilde{\gamma}''(s)| \, ds + \Big| \int_0^L \big(\tilde{\gamma}'(s) - \gamma'(s)\big) \, ds \Big| \\ &\leq \frac{1}{\sqrt{3}} L^{3/2} \Big(\int_0^L |\tilde{\gamma}''(s)|^2 \, ds \Big)^{1/2} + \int_0^L \big|\tilde{\gamma}'(s) - \gamma'(s)\big| \, ds. \end{aligned} \tag{7.92}$$

Summing the corresponding inequalities over $i \in J(\varepsilon)$ and using (7.83) and (7.84) we obtain with the Hölder inequality

$$\begin{aligned} \sum_{i \in J(\varepsilon)} L_{\varepsilon,i} &\leq C(\Lambda) \Big(\sum_{i \in J(\varepsilon)} L_{\varepsilon,i}^3 \Big)^{1/2} \Lambda + \varepsilon C(\Lambda) \\ &\leq \delta \Big(\sum_{i \in J(\varepsilon)} L_{\varepsilon,i} \Big) C(\Lambda) + \varepsilon C(\Lambda) \\ &\leq \frac{1}{2} \Big(\sum_{i \in J(\varepsilon)} L_{\varepsilon,i} \Big) + \varepsilon C(\Lambda) \end{aligned}$$

for $\delta = \delta(\Lambda)$ small enough. This proves (7.91). □

Proposition 7.22. *There exists a subsequence $\varepsilon \to 0$ and a $W^{2,2}$-system $\Gamma = \{c_i, i = 1, ..., N_1\}$ of closed curves without transversal crossings such that*

$$\varepsilon |\nabla u_\varepsilon| \;\to\; \mu_\Gamma \tag{7.93}$$

as measures on $\mathbb{R}^2$. Moreover Γ has even multiplicity and satisfies

$$\mathcal{W}(\Gamma) \;\leq\; 2 \liminf_{\varepsilon \to 0} \mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon). \tag{7.94}$$

Proof. By Lemma 7.21 we can, without loss of generality, assume that $L_{\varepsilon,i} > \delta(\Lambda)$ for all ε, i. We then deduce that the number of curves $N(\varepsilon)$ is bounded uniformly in ε. Next we consider the reparametrized one-periodic functions $c_{\varepsilon,i} : \mathbb{R} \to \mathbb{R}^2$ defined by

$$c_{\varepsilon,i}(r) \;:=\; \gamma_{\varepsilon,i}(L_{\varepsilon,i} r) \quad \text{for } r \in \mathbb{R}.$$

Since the functions $c_{\varepsilon,i}$ are Lipschitz continuous with uniformly bounded Lipschitz constant, and since $L_{\varepsilon,i}$ are bounded from above and away from zero, there exists a subsequence $\varepsilon \to 0$, a number $N_1 \in \mathbb{N}$ and

$$L_i > 0, \; c_i : \mathbb{R} \to \mathbb{R}^2, \quad i = 1, ..., N_1,$$

such that for all $i \in \{1, ..., N_1\}$ the functions c_i are Lipschitz continuous and

$$\begin{aligned} N(\varepsilon) &= N_1 \quad \text{for all } \varepsilon > 0, \\ L_{\varepsilon,i} &\to L_i \quad \text{as } \varepsilon \to 0, & (7.95)\\ c_{\varepsilon,i} &\to c_i \quad \text{in } C^{0,\sigma}(\mathbb{R};\mathbb{R}^2) \quad \text{as } \varepsilon \to 0 & (7.96)\end{aligned}$$

holds for all $0 < \sigma < 1$. In particular, the functions c_i are one-periodic and $\varepsilon|\nabla u_\varepsilon|$ converge as measures to μ_Γ.

We next turn to the $W^{2,2}$-bound on the limit curves c_i. We fix $0 < \lambda < 1$ and consider for $\gamma_{\varepsilon,i}$ modified functions $\tilde{\gamma}_{\varepsilon,i}$ as in Definition 7.18 and the reparametrizations $\tilde{c}_{\varepsilon,i}$,

$$\tilde{c}_{\varepsilon,i}(r) = \tilde{\gamma}_{\varepsilon,i}(L_{\varepsilon,i} r) \quad \text{for } r \in \mathbb{R}.$$

By Lemma 7.20 these curves are uniformly bounded in $W^{2,2}_{\text{loc}}(\mathbb{R};\mathbb{R}^2)$ and we deduce that there exists a subsequence $\varepsilon \to 0$ of the sequence in (7.96) such that the curves $\tilde{c}_{\varepsilon,i}$ converge weakly in $W^{2,2}_{\text{loc}}(\mathbb{R};\mathbb{R}^2)$ and strongly in $C^{1,\sigma}([-K,K];\mathbb{R}^2)$ for all $0 < \sigma < 1/2$, $K > 0$. Since $c_{\varepsilon,i}(0) = \tilde{c}_{\varepsilon,i}(0)$ we obtain for any $r \in \mathbb{R}$

$$\begin{aligned} \lim_{\varepsilon\to 0} |c_{\varepsilon,i}(r) - \tilde{c}_{\varepsilon,i}(r)| &\leq \lim_{\varepsilon\to 0} \Big| \int_0^r |c'_{\varepsilon,i}(\varrho) - \tilde{c}'_{\varepsilon,i}(\varrho)|\, d\varrho \Big| \\ &\leq \lim_{\varepsilon\to 0} \Big(1 + \frac{|r|}{L_{\varepsilon,i}}\Big) \int_0^{L_{\varepsilon,i}} |\gamma'_{\varepsilon,i}(s) - \tilde{\gamma}'_{\varepsilon,i}(s)|\, ds \\ &= 0, \end{aligned}$$

where we have used (7.83), the periodicity of $c_{\varepsilon,i}$, $\tilde{c}'_{\varepsilon,i}$ and $L_{\varepsilon,i} \geq \delta(\Lambda)$. Therefore we can identify the limits of $c_{\varepsilon,i}, \tilde{c}_{\varepsilon,i}$ and deduce that

$$\begin{aligned} \tilde{c}_{\varepsilon,i} &\to c_i \quad \text{weakly in } W^{2,2}_{\text{loc}}(\mathbb{R};\mathbb{R}^2) \quad \text{as } \varepsilon \to 0, & (7.97)\\ \tilde{c}_{\varepsilon,i} &\to c_i \quad \text{in } C^{1,\sigma}([-K,K];\mathbb{R}^2) \quad \text{as } \varepsilon \to 0, & (7.98)\end{aligned}$$

for all $0 \leq \sigma < 1/2$, $K > 0$. In particular $c_i \in W^{2,2}_{\text{loc}}(\mathbb{R};\mathbb{R}^2)$. The lower-semicontinuity of the norm under weak convergence and (7.85) yield that

$$\begin{aligned} \sum_{i=1}^{N_1} L_i^{1-2p} \int_0^1 c_i''(r)^p\, dr &\leq \liminf_{\varepsilon\to 0} \sum_{i=1}^{N_1} L_{\varepsilon,i}^{1-2p} \int_0^1 \tilde{c}''_{\varepsilon,i}(r)^p\, dr \\ &= \liminf_{\varepsilon\to 0} \sum_{i=1}^{N_1} \int_0^{L_{\varepsilon,i}} \tilde{\gamma}''_{\varepsilon,i}(s)^p\, ds \\ &\leq \Big[\sum_{i=1}^{N_1} L_i\Big]^{1-\frac{p}{2}} \Big[4(1-\lambda)^{-4} \liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon(u_\varepsilon, v_\varepsilon)\Big]^{\frac{p}{2}} \end{aligned}$$

holds for all $1 \leq p < 2$. Since $0 < \lambda < 1$ and $1 \leq p < 2$ are arbitrary we obtain (7.94).

Since the curves $c_{\varepsilon,i}$ are pairwise disjoint and have no self-intersections, by (7.96) we deduce that the curves c_i have no transversal crossings. Finally we obtain for

any $\eta \in C_c^1(\mathbb{R}^2)$ by the divergence theorem that

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0} \int_{\mathrm{supp}(u_\varepsilon)} \nabla \cdot \eta(x)\, dx \\ &= \lim_{\varepsilon\to 0} \sum_{i=1}^{N_1} \int_0^1 \eta(c_{\varepsilon,i}(s)) \cdot c_{\varepsilon,i}'(s)^\perp\, ds \\ &= \sum_{i=1}^{N_1} \int_0^1 \eta(c_i(s)) \cdot c_i'(s)^\perp\, ds \\ &= \int_{\mathrm{supp}(\Gamma)} \eta(x) \cdot \Big[\sum_{\{(i,s):c_i(s)=x\}} \frac{1}{L_i} c_i'(s)^\perp \Big]\, d\mathcal{H}^1(x). \end{aligned}$$

We deduce that

$$\sum_{(i,s):c_i(s)=x} \frac{1}{L_i} c_i'(s)^\perp \quad = \quad 0 \qquad \text{for } \mathcal{H}^1 - \text{almost all } x \in \mathrm{supp}(\Gamma)$$

and since $|c_i'| = L_i$ and the vectors $\{c_i'(s) \,:\, c_i(s) = x\}$ are collinear, this implies that $\theta_\Gamma(x) = \#\{(i,s) : c_i(s) = x\}$ is even. □

Using results from [13] we deduce that μ_Γ as in Proposition 7.22 is given by an alternative system of curves, where each curve is passed twice.

Lemma 7.23. *Let $\Gamma = \{c_i, i = 1, ..., N\}$ be a $W^{2,2}$-system of closed curves without transversal crossings and with an even multiplicity function θ_Γ. Then there exists a system of curves $\overline{\Gamma} = \{\overline{\gamma}_i : i = 1, ..., \overline{N}\}$ such that*

$$\mu_\Gamma = 2\mu_{\overline{\Gamma}}, \tag{7.99}$$

$$\mathcal{W}(\Gamma) = 2\mathcal{W}(\overline{\Gamma}), \tag{7.100}$$

$$\theta_\Gamma = 2\theta_{\overline{\Gamma}}. \tag{7.101}$$

Proof. This Lemma requires some machinery from geometric measure theory. For the relevant definitions of Sobolev type submanifolds, Hutchinson varifolds and the various definitions for tangential lines and multiplicity functions see [13].

By [13] Remark 4.9 we obtain that

$$f := \theta(\Gamma, \cdot)\mathcal{X}_{\mathrm{supp}(\Gamma)}$$

belongs to $\mathcal{S}^2_{tg}(\mathbb{R}^2)$, that is the set of Sobolev-type submanifold with the additional property that a unique tangential line exists in all points of the support. By [13, Proposition 4.5] this implies that $f \in HV^2(\mathbb{R}^2)$ is a Hutchinson varifold such that a unique tangential line exists in all points of the support. Since the multiplicity is even we conclude that also $\frac{1}{2}f$ belongs to $HV^2(\mathbb{R}^2)$ and that a unique tangential line exists in all points of the support. Using again [13, Proposition 4.5] we obtain that $\frac{1}{2}f \in \mathcal{S}^2_{tg}(\mathbb{R}^2)$. Now [13, Corollary 4.8] gives that $\frac{1}{2}f$ is given by a $W^{2,2}$-system $\overline{\Gamma} = \{\overline{\gamma}_i : i = 1, ..., \overline{N}\}$ of closed curves without transversal crossings. This implies (7.99). Equations (7.100) and (7.101) are an immediate consequence. □

We next show that the limit of the boundaries of the support of u_ε and the limit of the mass distributions u_ε are identical.

Proposition 7.24. *For a system of curves* Γ *as obtained in Proposition 7.22 and the measure* μ *in* (4.3) *we have*

$$\mu_\Gamma \,=\, \mu.$$

In particular, (7.93) *holds for the whole sequence* $\varepsilon \to 0$.

Proof. Let $\partial\,\mathrm{supp}(u_\varepsilon)$ be given by a system of closed curves $\Gamma_\varepsilon = \{\gamma_{\varepsilon,i}\}_{i=1,\dots,N(\varepsilon)}$ as in Remark 7.2. defined in Definition 7.3 we obtain from Lemma 7.10 that for all $\eta \in C^1_c(\mathbb{R}^2; R^+_0)$

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^2} \eta u_\varepsilon - \int_{\mathbb{R}^2} \eta\varepsilon|\nabla u_\varepsilon|\,\Big|\\
=&\Big|\sum_{i=1}^{N(\varepsilon)} \int_0^{L_{\varepsilon,i}} \Big(\int_0^{M_{\varepsilon,i}(s)} \eta\big(\gamma_{\varepsilon,i}(s) + \mathsf{t}_{\varepsilon,i}(s,m)\theta_{\varepsilon,i}(s)\big)\,dm - \eta\big(\gamma_{\varepsilon,i}(s)\big)\Big)\,ds\,\Big|\\
\leq& \sum_{i=1}^{N(\varepsilon)} \int_0^{L_{\varepsilon,i}} \big|M_{\varepsilon,i}(s) - 1\big|\eta\big(\gamma_{\varepsilon,i}(s)\big)\,ds\\
&+ \sum_{i=1}^{N(\varepsilon)} \int_0^{L_{\varepsilon,i}} \int_0^{M_{\varepsilon,i}(s)} \Big|\eta\big(\gamma_{\varepsilon,i}(s) + \mathsf{t}_{\varepsilon,i}(s,m)\theta_{\varepsilon,i}(s)\big) - \eta\big(\gamma_{\varepsilon,i}(s)\big)\Big|\,dm\,ds\\
\leq& \Big(\sum_{i=1}^{N(\varepsilon)} \int_0^{L_{\varepsilon,i}} \big(M_{\varepsilon,i}(s) - 1\big)^2\,ds\Big)^{\frac12}\Big(\sum_{i=1}^{N(\varepsilon)} \int_0^{L_{\varepsilon,i}} \eta\big(\gamma_{\varepsilon,i}(s)\big)^2\,ds\Big)^{\frac12}\\
&+ \|\eta\|_{C^1(\mathbb{R}^2)} \sum_{i=1}^{N(\varepsilon)} \int_0^{L_{\varepsilon,i}} \int_0^{M_{\varepsilon,i}(s)} \mathsf{t}_{\varepsilon,i}(s,m)\,dm\,ds\\
\leq& \,\varepsilon\mathcal{G}_\varepsilon(u_\varepsilon,v_\varepsilon)^{\frac12} 2\|\eta\|_{C^0(\mathbb{R}^2)} + \|\eta\|_{C^1(\mathbb{R}^2)} d_1(u_\varepsilon,v_\varepsilon),
\end{aligned} \tag{7.102}$$

where we have used (7.71). We observe that by (7.2)

$$\frac{1}{\varepsilon} d_1(u_\varepsilon,v_\varepsilon) \,\leq\, 2 + \varepsilon^2\Lambda$$

and we deduce from (7.102) that

$$\Big|\int_{\mathbb{R}^2} \eta u_\varepsilon - \int_{\mathbb{R}^2} \eta\varepsilon|\nabla u_\varepsilon|\,\Big| \,\leq\, \varepsilon\Big(2\Lambda^{\frac12}\|\eta\|_{C^0(\mathbb{R}^2)} + (2+\varepsilon^2\Lambda)\|\eta\|_{C^1(\mathbb{R}^2)}\Big)$$

This shows that

$$\mu \,=\, \lim_{\varepsilon\to 0} u_\varepsilon\mathcal{L}^2 \,=\, \lim_{\varepsilon\to 0} \varepsilon|\nabla u_\varepsilon| \,=\, \mu_\Gamma$$

in the sense of convergence as measures. □

The proof of the lim-inf estimate now follows: by Proposition 7.22 there exists a system of curves Γ of even multiplicity such that

$$\frac12 \mathcal{W}(\Gamma) \leq \liminf_{\varepsilon\to 0} \mathcal{G}_\varepsilon(u_\varepsilon,v_\varepsilon).$$

By Lemma 7.23 there is an alternative system of curves $\tilde\Gamma$, of integer multiplicity, such that $\mu_\Gamma = 2\mu_{\tilde\Gamma}$ and $\mathcal{W}(\Gamma) = 2\mathcal{W}(\tilde\Gamma)$. This second system $\tilde\Gamma$ is the system referred to in Theorem 4.1 as Γ. Finally, Proposition 7.24 guarantees that $\mu_\Gamma = 2\mu_{\tilde\Gamma}$ is the limit of the sequence $(u_\varepsilon, v_\varepsilon)$ in the appropriate sense (see (4.3)).

8. The limsup estimate

8.1. **The heart of the construction.** Let us first perform the lim-sup construction in the case that $\mu = 2\mu_\Gamma$, where Γ consists of one simple curve.

Lemma 8.1. *Let $\gamma : [0, L) \to \mathbb{R}^2$ be a closed C^2-curve without self-intersections and with $|\gamma'| = 1$ and let $\Gamma = \{\gamma\}$. Then there exists $\varepsilon_0 > 0$, $\varepsilon_0 = \varepsilon_0(\|\gamma''\|_{C^0([0,L))})$ and sequences $(u_\varepsilon, v_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ with*

$$u_\varepsilon, v_\varepsilon \in BV(\mathbb{R}^2; \{0, 1/\varepsilon\}), \quad u_\varepsilon v_\varepsilon = 0, \tag{8.1}$$

$$\int_{\mathbb{R}^2} u_\varepsilon = \int_{\mathbb{R}^2} v_\varepsilon = 2L, \tag{8.2}$$

$$\operatorname{supp}(u_\varepsilon) \cup \operatorname{supp}(v_\varepsilon) \subset \{x : \operatorname{dist}(x, \Gamma) \le 3\varepsilon\}, \tag{8.3}$$

such that

$$2\mu_\Gamma = \lim_{\varepsilon \to 0} u_\varepsilon \mathcal{L}^2, \tag{8.4}$$

$$\frac{1}{2} \int_0^L \gamma''(s)^2 \, ds \ge \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \Big(\frac{1}{\varepsilon} d_1(u_\varepsilon, v_\varepsilon) + \varepsilon \int_{\mathbb{R}^2} |\nabla u_\varepsilon| - 4L \Big). \tag{8.5}$$

Proof. There exists a $\varepsilon_0 > 0$ such that

$$\sup_{s \in [0,L)} |\gamma''(s)| \le \frac{1}{4\varepsilon_0}, \tag{8.6}$$

$$x \mapsto \operatorname{dist}(x, \Gamma) \text{ is of class } C^2 \text{ in } \{x : \operatorname{dist}(x, \Gamma) < 4\varepsilon_0\}. \tag{8.7}$$

Let $\nu : [0, L) \to S^1$ be the C^1 unit normal field of γ that satisfies

$$\det(\gamma', \nu) = -1$$

and let $\kappa(s)$ be the curvature in direction of $\nu(s)$,

$$\kappa(s) = -\nu'(s) \cdot \gamma'(s) = \nu(s) \cdot \gamma''(s).$$

For $\varepsilon < \varepsilon_0$ we set

$$u_\varepsilon(x) := \begin{cases} \frac{1}{\varepsilon} & \text{if } \operatorname{dist}(x, \Gamma) < \varepsilon, \\ 0 & \text{elsewise}, \end{cases}$$

and define two curves $\hat{\gamma}_+, \hat{\gamma}_- : [0, L) \to \mathbb{R}^2$ by

$$\begin{aligned} \hat{\gamma}_+(s) &:= \gamma(s) + \varepsilon \nu(s), \\ \hat{\gamma}_-(s) &:= \gamma(s) - \varepsilon \nu(s), \end{aligned}$$

see Figure 8. Using the parametrization $\Phi(s, r) := \gamma(s) + r\nu(s)$ we calculate that

$$|\det D\Phi(s, r)| = 1 - r\kappa(s) \quad \text{for } 0 < s < L, \ -\varepsilon < r < \varepsilon.$$

Therefore for any $\eta \in C^0(\mathbb{R}^2)$

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \eta(x) u_\varepsilon(x) \, ds &= \lim_{\varepsilon \to 0} \int_0^L \int_{-\varepsilon}^{\varepsilon} \eta(\Phi(s, r)) \frac{1}{\varepsilon} (1 - r\kappa(s)) \, dr \, ds \\ &= \int_0^L 2\eta(\Phi(s, 0)) \, ds = 2 \int_0^L \eta(\gamma(s)) \, ds \end{aligned}$$

holds and we deduce (8.4).

The arc length functions corresponding to $\hat\gamma_+, \hat\gamma_-$ are given by

$$\begin{aligned}\vartheta_+(s) &= \int_0^s |\hat\gamma_+'(\sigma)|\,d\sigma \;=\; \int_0^s (1-\varepsilon\kappa(\sigma))\,d\sigma,\\ \vartheta_-(s) &= \int_0^s |\hat\gamma_-'(\sigma)|\,d\sigma \;=\; \int_0^s (1+\varepsilon\kappa(\sigma))\,d\sigma.\end{aligned}$$

Let $\tilde\gamma_+, \tilde\gamma_-$ be the corresponding parametrizations with respect to arclength,

$$\begin{aligned}\tilde\gamma_+ &= \hat\gamma_+\circ\vartheta_+^{-1}, &\quad \tilde\gamma_+ : [0,L_+)\to\mathbb{R}^2, \quad L_+ &= L-\varepsilon\int_0^L \kappa(\sigma)\,d\sigma,\\ \tilde\gamma_- &= \hat\gamma_-\circ\vartheta_-^{-1}, &\quad \tilde\gamma_- : [0,L_-)\to\mathbb{R}^2, \quad L_- &= L+\varepsilon\int_0^L \kappa(\sigma)\,d\sigma.\end{aligned}$$

The curves $\tilde\gamma_+, \tilde\gamma_-$ parametrize the boundary of $\{u_\varepsilon = 1/\varepsilon\}$ and we obtain that

$$\int_{\mathbb{R}^2} \varepsilon|\nabla u_\varepsilon| \;=\; L_+ + L_- \;=\; 2L. \tag{8.8}$$

Corresponding to $\tilde\gamma_+, \tilde\gamma_-$ we define unit normal fields

$$\begin{aligned}\tilde\nu_+ &:= \nu\circ\vartheta_+^{-1},\\ \tilde\nu_- &:= \nu\circ\vartheta_-^{-1}\end{aligned}$$

and we compute the curvature of $\tilde\gamma_+, \tilde\gamma_-$ in direction of $\nu_\pm$

$$\begin{aligned}\tilde\kappa_+ &= \frac{\kappa\circ\vartheta_+^{-1}}{1-\varepsilon\kappa\circ\vartheta_+^{-1}},\\ \tilde\kappa_- &= \frac{\kappa\circ\vartheta_-^{-1}}{1+\varepsilon\kappa\circ\vartheta_-^{-1}}.\end{aligned}$$

Observe that by (8.6) for $\varepsilon < \varepsilon_0$

$$|\tilde\kappa_+|, |\tilde\kappa_-| \;\le\; \frac{1}{3\varepsilon_0}. \tag{8.9}$$

As in subsection 7.1 we define *mass coordinates* $\mathfrak{m}_+, \mathfrak{m}_-$ by

$$\begin{aligned}\mathfrak{m}_+(r,t) &:= \frac{1}{\varepsilon}\big(t-\frac{t^2}{2}\tilde\kappa_+(r)\big),\\ \mathfrak{m}_-(r,t) &:= \frac{1}{\varepsilon}\big(t-\frac{t^2}{2}\tilde\kappa_-(r)\big)\end{aligned}$$

and the inverse mappings $\mathfrak{t}_+(r,\cdot) = \mathfrak{m}_+(r,\cdot)^{-1}$, $\mathfrak{t}_-(r,\cdot) = \mathfrak{m}_-(r,\cdot)^{-1}$,

$$\begin{aligned}\mathfrak{t}_+(r,m) &= \frac{1}{\tilde\kappa_+(r)}\Big(1-\sqrt{1-2\varepsilon\tilde\kappa_+ m}\Big),\\ \mathfrak{t}_-(r,m) &= \frac{1}{\tilde\kappa_-(r)}\Big(1-\sqrt{1-2\varepsilon\tilde\kappa_- m}\Big).\end{aligned}$$

By (8.9) the expressions $1-2\varepsilon\tilde\kappa_\pm m$ are positive for $|m|\le 1$. Using that

$$\begin{aligned}\sqrt{1+2z} &\le 1+z, &\quad &\text{for } z\ge -\frac{1}{2},\\ \sqrt{1+2z} &\ge 1+2z, &\quad &\text{for } -\frac{1}{2}\le z\le 0\end{aligned}$$

we deduce that

$$0 \le \frac{1}{m}\mathfrak{t}_+(r,m) \le 2\varepsilon, \qquad 0 \le \frac{1}{m}\mathfrak{t}_-(r,m) \le 2\varepsilon. \tag{8.10}$$

We construct parametrizations ψ_+, ψ_- in the same way as we did in subsection 7.1. Considering $\tilde{\gamma}_+$, the unit normal field $\tilde{\nu}_+$ takes the role of the direction field θ in subsection 7.1. Therefore $\tilde{\kappa}_+$ and 1 appear in place of α' and $\sin\beta$. Note that, with respect to $\tilde{\gamma}_+$, $\mathrm{supp}(u_\varepsilon)$ is in the direction of $-\tilde{\nu}_+$ and, with respect to $\tilde{\gamma}_-$, in direction of $\tilde{\nu}_-$.

For $\tilde{\gamma}_+, \tilde{\gamma}_-$ respectively we define 'ray lengths' $l_+(r), l_-(r)$ such that their values

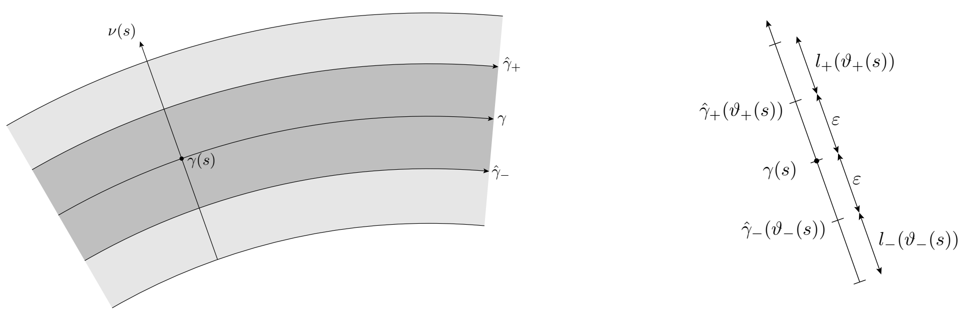

Figure 8. The limsup construction.

in mass coordinates become -1 and 1,

$$l_+(r) := \mathfrak{t}_+(r,-1) = \frac{1}{\tilde{\kappa}_+(r)}\Big(1 - \sqrt{1 + 2\varepsilon\tilde{\kappa}_+}\Big), \tag{8.11}$$

$$l_-(r) := \mathfrak{t}_-(r,1) = \frac{1}{\tilde{\kappa}_-(r)}\Big(1 - \sqrt{1 - 2\varepsilon\tilde{\kappa}_-}\Big). \tag{8.12}$$

To parametrize $\mathrm{supp}(u_\varepsilon)$ we define sets

$$D_+ := \{(r,t) : 0 \le r < L_+, l_+(r) < t < 0\}$$
$$D_- := \{(r,t) : 0 \le r < L_-, 0 < t < l_-(r)\}.$$

and the maps

$$\psi_+(r,t) := \tilde{\gamma}_+(r) + t\tilde{\nu}_+(r) \qquad \text{for } 0 < r < L_+, -2\varepsilon < t < 2\varepsilon$$
$$\psi_-(r,t) := \tilde{\gamma}_-(r) + t\tilde{\nu}_-(r) \qquad \text{for } 0 < r < L_-, -2\varepsilon < t < 2\varepsilon,$$

which are, for $\varepsilon < \varepsilon_0$, by (8.7) injective on their domains. Analogous to Lemma 7.12 we observe that for any $\eta \in C^0_c(\mathbb{R}^2)$

$$\int_{\psi_+(D_+)} \eta(x)u_\varepsilon(x)\,dx = \int_0^{L_+}\int_{-1}^0 \eta\big(\psi_+(r,\mathfrak{t}_+(r,m))\big)\,dm\,dr, \tag{8.13}$$

$$\int_{\psi_-(D_-)} \eta(x)u_\varepsilon(x)\,dx = \int_0^{L_-}\int_0^1 \eta\big(\psi_-(r,\mathfrak{t}_-(r,m))\big)\,dm\,dr. \tag{8.14}$$

Moreover we deduce from (8.11), (8.12) that

$$l_+(\vartheta_+(s)) = \frac{1}{\kappa(s)}\Big(1 - \varepsilon\kappa(s) - \sqrt{1-(\varepsilon\kappa(s))^2}\Big),$$
$$l_-(\vartheta_-(s)) = \frac{1}{\kappa(s)}\Big(1 + \varepsilon\kappa(s) - \sqrt{1-(\varepsilon\kappa(s))^2}\Big).$$

which implies that

$$l_-(\vartheta_-(s)) - l_+(\vartheta_+(s)) \;=\; 2\varepsilon.$$

We conclude that the sets $\psi_+(D_+)$ and $\psi_-(D_-)$ are pairwise disjoint and cover almost all of $\operatorname{supp}(u_\varepsilon)$. We therefore obtain by (8.13), (8.14) that

$$\int \eta(x)u_\varepsilon(x)\,dx$$
$$= \int_0^{L_+}\int_{-1}^{0} \eta\big(\psi_+(r,\mathfrak{t}_+(r,m))\big)\,dm\,dr + \int_0^{L_-}\int_0^1 \eta\big(\psi_-(r,\mathfrak{t}_-(r,m))\big)\,dm\,dr$$

holds. In particular we obtain

$$\int_{\mathbb{R}^2} u_\varepsilon \;=\; L_+ + L_- \;=\; 2L. \tag{8.15}$$

We now define an injective transport map $S : \operatorname{supp}(u_\varepsilon) \to \mathbb{R}^2$ by

$$S(\psi_+(r,t)) \;:= \psi_+\big(r,\mathfrak{t}_+\big(r,\mathfrak{m}_+(r,t)+1\big)\big) \text{ for } (r,t) \in D_+,$$
$$S(\psi_-(r,t)) \;:= \psi_-\big(r,\mathfrak{t}_-\big(r,\mathfrak{m}_-(r,t)-1\big)\big) \text{ for } (r,t) \in D_-$$

and set

$$v_\varepsilon \;:=\; \frac{1}{\varepsilon}\mathcal{X}_{S(\operatorname{supp}(u_\varepsilon))}.$$

This gives

$$\operatorname{supp}(v_\varepsilon) \;=\; \{\psi_+(r,t) : 0 \le t \le \mathfrak{t}_+(r,1)\} \cup \{\psi_-(r,t) : \mathfrak{t}_-(r,-1) \le t \le 0\}.$$

By (8.10) we obtain that $\mathfrak{t}_+(r,1), |\mathfrak{t}_-(r,-1)| \le 2\varepsilon$ and since the distance of $\hat{\gamma}_+$ and $\hat{\gamma}_-$ to γ equals ε the equation (8.3) follows. We compute for an arbitrary $\eta \in C^0_c(\mathbb{R}^2)$ that

$$\int \eta(S(x))u_\varepsilon(x)\,dx$$
$$= \int_0^{L_+}\int_{-1}^{0} \eta\big(S(\psi_+(r,\mathfrak{t}_+(r,m)))\big)\,dm\,dr + \int_0^{L_-}\int_0^1 \eta\big(S(\psi_-(r,\mathfrak{t}_-(r,m)))\big)\,dm\,dr$$
$$= \int_0^{L_+}\int_{-1}^{0} \eta\big(\psi_+(r,\mathfrak{t}_+(r,m+1))\big)\,dm\,dr + \int_0^{L_-}\int_0^1 \eta\big(\psi_-(r,\mathfrak{t}_-(r,m-1))\big)\,dm\,dr$$
$$= \int_0^{L_+}\int_0^{1} \eta\big(\psi_+(r,\mathfrak{t}_+(r,m))\big)\,dm\,dr + \int_0^{L_-}\int_{-1}^0 \eta\big(\psi_-(r,\mathfrak{t}_-(r,m))\big)\,dm\,dr$$
$$= \int \eta(x)v_\varepsilon(x)\,dx,$$

where we have used the change-of-variables formula in the last equality. This shows that S indeed is a transport map. In particular it follows from (8.15) that

$\int_{\mathbb{R}^2} v_\varepsilon = 2L$ holds, which proves (8.2). Moreover we obtain

$$\begin{aligned} d_1(u_\varepsilon, v_\varepsilon) &\leq \int_{\mathbb{R}^2} |x - S(x)| u_\varepsilon(x)\,dx \\ &= \int_0^{L_+} \int_{-1}^{0} \big(\mathfrak{t}_+(r, m+1) - \mathfrak{t}_+(r, m)\big)\,dm\,dr \\ &\quad + \int_0^{L_-} \int_0^1 \big(\mathfrak{t}_-(r, m) - \mathfrak{t}_-(r, m-1)\big)\big)\,dm\,dr \end{aligned} \tag{8.16}$$

By (8.11) we have $\mathfrak{m}_+(r, l_+(r)) = -1$ and we deduce for the inner integral of the first term on the right-hand side of (8.16) that

$$\begin{aligned} &\int_{-1}^{0} \big(\mathfrak{t}_+(r, m+1) - \mathfrak{t}_+(r, m)\big)\,dm \\ &= \int_0^{\mathfrak{m}_+(r, l_+(r))} \mathfrak{t}_+(r, m) - \mathfrak{t}_+\big(r, m - \mathfrak{m}_+(r, l_+(r))\big)\,dm. \end{aligned}$$

Therefore Lemma 7.16 yields that

$$\int_{-1}^{0} \big(\mathfrak{t}_+(r, m+1) - \mathfrak{t}_+(r, m)\big)\,dm \;=\; \varepsilon + \frac{1}{4}\varepsilon^3 \tilde{\kappa}_+(r)^2 + R_+(r)\varepsilon^5, \tag{8.17}$$

where

$$0 \;\leq\; R_+(r) \;\leq\; \tilde{\kappa}_+(r)^4 \;\leq\; C(\varepsilon_0) \tag{8.18}$$

by (7.73) and (8.6). For the inner integral of the second term in (8.16) we deduce by similar arguments

$$\int_0^1 \mathfrak{t}_-(r, m) - \mathfrak{t}_-(r, m-1)\,dm = \varepsilon + \frac{1}{4}\varepsilon^3 \tilde{\kappa}_-(r)^2 + R_-(r)\varepsilon^5, \tag{8.19}$$

with

$$0 \;\leq\; R_-(r) \;\leq\; \tilde{\kappa}_-(r)^4 \;<\; C(\varepsilon_0). \tag{8.20}$$

By (8.16) and (8.17)-(8.20) we obtain that

$$\begin{aligned} &\frac{1}{\varepsilon} d_1(u_\varepsilon, v_\varepsilon) \\ &\leq L_+ + L_- + \frac{1}{4}\varepsilon^2 \int_0^{L_+} \tilde{\kappa}_+(r)^2\,dr + \frac{1}{4}\varepsilon^2 \int_0^{L_-} \tilde{\kappa}_-(r)^2\,dr + \varepsilon^4 L C(\varepsilon_0). \end{aligned} \tag{8.21}$$

We compute that

$$\begin{aligned} \int_0^{L_+} \tilde{\kappa}_+(r)^2\,dr + \int_0^{L_-} \tilde{\kappa}_-(r)^2\,dr \;&=\; \int_0^L \Big(\frac{\kappa(s)^2}{1 - \varepsilon\kappa(s)} + \frac{\kappa(s)^2}{1 + \varepsilon\kappa(s)}\Big)\,ds \\ &=\; 2\int_0^L \kappa(s)^2\,ds + 2\varepsilon^2 \int_0^L \frac{\kappa(s)^2}{1 - \varepsilon^2\kappa(s)^2}\,ds \\ &\leq\; 2\int_0^L \kappa(s)^2\,ds + 2\varepsilon^2 L C(\varepsilon_0) \end{aligned}$$

which gives with (8.21) and $2L = L_+ + L_-$ that

$$\frac{1}{\varepsilon} d_1(u_\varepsilon, v_\varepsilon) \;\leq\; 2L + \frac{\varepsilon^2}{2} \int_0^L \kappa(s)^2\,ds + \varepsilon^4 L C(\varepsilon_0). \tag{8.22}$$

Since $|\kappa| = |\gamma''|$ we obtain from (8.8) and (8.22) that

$$\frac{1}{\varepsilon^2}\Big(\frac{1}{\varepsilon}d_1(u_\varepsilon, v_\varepsilon) + \varepsilon\int_{\mathbb{R}^2}|\nabla u_\varepsilon| - 4L\Big) \;\leq\; \frac{1}{2}\int_0^L \gamma''(s)^2\,ds + \varepsilon^2 LC(\varepsilon_0),$$

which proves (8.5). □

8.2. **The general case.** We prove now the limsup estimate stated in Theorem 4.1. First we need a technical Lemma which follows from results in [11] and [12] and approximates a given system of curves as in Theorem 4.1 strongly in $W^{2,2}$ by a sequence of more regular systems of curves.

Lemma 8.2. *Let Γ be a $W^{2,2}$-system of closed curves without transversal crossings. Then there exists a sequence $(\Gamma^j)_{j\in\mathbb{N}}$ of $W^{2,2}$-systems of closed curves and a number $m \in \mathbb{N}$ such that the following holds:*

(a) For all $j \in \mathbb{N}$ the system of curves $\Gamma^j = \{\gamma_k^j\}_{k=1,\dots,m}$ is given by a pairwise disjoint family of simple closed curves and satisfies

$$|\Gamma^j| = |\Gamma|. \tag{8.23}$$

(b) For all $1 \leq k \leq m$ holds

$$\gamma_k^j \;\to\; \gamma_k \quad \text{ in } W^{2,2}(0,1) \quad \text{ as } j \to \infty. \tag{8.24}$$

(c) $\tilde\Gamma := \{\gamma_k\}_{k=1,\dots,m}$ is a $W^{2,2}$-system of closed curves without transversal crossings.
(d) $\tilde\Gamma$ is equivalent to Γ in the sense that

$$\mu_\Gamma \;=\; \mu_{\tilde\Gamma}.$$

In particular we have

$$\mathcal{W}(\Gamma^j) \;\to \mathcal{W}(\Gamma) \quad \text{ as } j \to \infty. \tag{8.25}$$

Proof. By [12, Corollary 5.2] there exists a sequence $(\tilde\Gamma^j)_{j\in\mathbb{N}}$ and a $W^{2,2}$-system of closed curves $\tilde\Gamma$ such that

(1) for all $j \in \mathbb{N}$ $\tilde\Gamma^j$ is an oriented parametrization of a bounded open smooth set $E_j \subset \mathbb{R}^2$,
(2) $\tilde\Gamma^j$ converge weakly in $W^{2,2}$ to $\tilde\Gamma$ as $j \to \infty$,
(3) $\tilde\Gamma^j$ converge 'in energy' to $\tilde\Gamma$ as $j \to \infty$.
(4) $\tilde\Gamma$ and Γ are equivalent.

Property (1) implies that $\tilde\Gamma^j$ is a disjoint system of simple closed curves parametrized on $(0,1)$ proportional to arclength.

By property (2) there exists a number $m \in \mathbb{N}$ such that

$$\tilde\Gamma^j \;=\; \{\tilde\gamma_k^j\}_{k=1,\dots,m}, \quad \tilde\Gamma \;=\; \{\tilde\gamma_k\}_{k=1,\dots,m},$$
$$\tilde\gamma_k^j,\, \tilde\gamma_k : (0,1) \;\to\; \mathbb{R}^2,$$
$$|(\tilde\gamma_k^j)'| \;=\; L_k^j, \quad |(\tilde\gamma_k)'| \;=\; L_k \text{ on } (0,1),$$

with

$$\tilde\gamma_k^j \;\to\; \tilde\gamma_k \quad \text{ weakly in } W^{2,2}(0,1) \text{ as } j \to \infty. \tag{8.26}$$

In particular we deduce that $L_k^j \to L_k$ as $j \to \infty$ for all $k = 1, ..., m$ and

$$|\tilde{\Gamma}^j| \to |\tilde{\Gamma}| \tag{8.27}$$

holds.

The 'convergence in energy' stated in (3) gives that, as $j \to \infty$,

$$|\tilde{\Gamma}^j| + \sum_{k=1}^m \int_0^1 (L_k^j)^{-3} |(\tilde{\gamma}_k^j)''|^2 \to |\tilde{\Gamma}| + \sum_{k=1}^m \int_0^1 (L_k)^{-3} |(\tilde{\gamma}_k)''|^2. \tag{8.28}$$

Together with (8.27) and the weak convergence (8.26) this yields

$$\tilde{\gamma}_k^j \to \tilde{\gamma}_k \quad \text{strongly in } W^{2,2}(0,1) \text{ as } j \to \infty, \tag{8.29}$$

$$\mathcal{W}(\tilde{\Gamma}^j) \to \mathcal{W}(\Gamma) \text{ as } j \to \infty. \tag{8.30}$$

Finally we define for $j \in \mathbb{N}$ a modified system of curves $\Gamma_k^j = \{\gamma_k^j\}_{k=1,...,m}$ by

$$\tilde{\gamma}_k^j(s) := \frac{|\Gamma|}{|\tilde{\Gamma}^j|} \tilde{\gamma}_k^j(s).$$

Then (8.23) is satisfied. By (8.27) we see that $\tilde{\gamma}_k^j$ and γ_k^j have for all $1 \leq k \leq m$ the same limits as $j \to \infty$. We therefore obtain from (8.29), (8.30) that (8.24), (8.25) hold. □

Proof of Theorem 4.1 (lim-sup part):

Let Γ be given as in Theorem 4.1 and $(\Gamma^j)_{j\in\mathbb{N}}$ be a sequence of systems of closed curves as in Lemma 8.2, given by $\Gamma^j = \{\gamma_k^j\}_{k=1,...,m}$. We now parametrize the curves γ_k^j by arclength,

$$\gamma_k^j : (0, L_k^j) \to \mathbb{R}^2, \quad |(\gamma_k^j)'| = 1.$$

Choose ε_0^j such that the distance functions $\mathrm{dist}(\cdot, \Gamma^j)$ is smooth in the set $\{x : \mathrm{dist}(x, \Gamma^j) < 4\varepsilon_0^j\}$.

For $0 < \varepsilon < \varepsilon_0^j$ we let $u_\varepsilon^{j,k}$, $v_\varepsilon^{j,k}$ be the approximations constructed in Lemma 8.1 and define

$$u_\varepsilon^j := \sum_{k=1}^m u_\varepsilon^{j,k}, \qquad v_\varepsilon^j := \sum_{k=1}^m v_\varepsilon^{j,k}.$$

By (8.3) all the functions $u_\varepsilon^{j,k}, v_\varepsilon^{j,k}$ have pairwise disjoint supports. We compute that

$$\int_{\mathbb{R}^2} u_\varepsilon^j \, d\mathcal{L}^2 = \sum_{k=1}^m \int_{\mathbb{R}^2} u_\varepsilon^{j,k} \, d\mathcal{L}^2 = \sum_{k=1}^m 2L_j^k = 2|\Gamma^j| = 2|\Gamma| = M$$

and, by similar calculations, $\int v_\varepsilon^j = M$. Moreover, by (8.4), (8.24),

$$\lim_{j\to\infty} \lim_{\varepsilon\to 0} u_\varepsilon^j \mathcal{L}^2 = \lim_{j\to\infty} \sum_{k=1}^m \lim_{\varepsilon\to 0} u_\varepsilon^{j,k} \mathcal{L}^2 = 2 \lim_{j\to\infty} \mu_{\Gamma^j} = 2\mu_\Gamma$$

and, by (8.25), (8.5),

$$\begin{aligned}
\mathcal{W}(\Gamma) &= \lim_{j\to\infty} \mathcal{W}(\Gamma^j) \\
&= \lim_{j\to\infty} \frac{1}{2}\sum_{k=1}^{m} \int_0^{L_k^j} \big|(\gamma_k^j)''\big|^2\, ds \\
&\geq \limsup_{j\to\infty} \sum_{k=1}^{m} \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\Big(\frac{1}{\varepsilon} d_1(u_\varepsilon^{j,k}, v_\varepsilon^{j,k}) + \varepsilon \int |\nabla u_\varepsilon^{j,k}| - 2\int u_\varepsilon^{j,k}\, d\mathcal{L}^2\Big) \\
&\geq \limsup_{j\to\infty} \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\Big(\frac{1}{\varepsilon} d_1(u_\varepsilon^{j}, v_\varepsilon^{j}) + \varepsilon \int |\nabla u_\varepsilon^{j}| - 2\int u_\varepsilon^{j}\, d\mathcal{L}^2\Big).
\end{aligned}$$

The final inequality follows from constructing an admissible joint transport map for $(u_\varepsilon^j, v_\varepsilon^j)$ from the transport maps of $(u_\varepsilon^{j,k}, v_\varepsilon^{j,k})$.

Therefore there exist subsequences $(j_l)_{l\in\mathbb{N}}$, $(\varepsilon_l)_{l\in\mathbb{N}}$ such that

$$\lim_{l\to\infty} u_{\varepsilon_l}^{j_l}\mathcal{L}^2 \;=\; 2\mu_\Gamma$$

as Radon measures on $\mathbb{R}^2$ and

$$\mathcal{W}(\Gamma) \;\geq\; \limsup_{l\to\infty} \mathcal{G}_{\varepsilon_l}(u_{\varepsilon_l}, v_{\varepsilon_l}).$$

This proves the lim-sup estimate in Theorem 4.1. □

9. Discussion

9.1. **General remarks.** In the Introduction we formulated two basic questions about self-aggregating, partially localized structures: first, why do they exist at all (and why are they stable), and secondly, how can we understand their resistance to stretching, bending, and fracture, that is observed experimentally.

The fact that the functional $\mathcal{F}_\varepsilon$ favours structures in which u and v alternate on a length scale ε is well illustrated by the special cases of Section 2. Simply put, for $d_1(u,v)$ to be moderate, the masses represented by u and v have to be close; the fact that $\|u\|_{L^\infty}$ and $\|v\|_{L^\infty}$ are limited implies that these masses occupy a certain amount of volume; and finally the interface penalization prevents the masses of u and v from being too finely interspersed. Together these restrictions impose a length scale on structures with moderate $\mathcal{F}_\varepsilon$.

This argument does not determine, however, the *geometry* of structures with moderate (or even lowest) energy. The fact that the straight lamellar geometry has lowest energy is not obvious, as is demonstrated by the ‘wriggled lamellar’ patterns of Ren and Wei in a strongly related system [42]. The reason for the penalization of curvature in the current context can be traced back to a property of strict convexity of the function $\mathfrak{t}(s,m)$ defined in (7.54). Since this convexity property lies at the heart of the stability of these partially localized structures, we take some time to investigate it further.

9.2. **The geometric basis for stability.** An essential observation in the proof is that it is more convenient to use *mass* coordinates along rays instead of *length* coordinates, since the Monge-Kantorovich distance measures spatial distances between pairs of infinitesimal portions of mass. The function $\mathfrak{t}(s,m)$ (see (7.54)) connects the two descriptions by giving the length coordinate t as a function of mass coordinate m.

For fixed m, the value of $\mathfrak{t}(s,m)$ depends on the value of $\alpha'(s)$, as is obvious both from the definition and from a geometric point of view (see Fig. 9 (left)). For the purpose of this discussion we write the dependence on $\alpha'(s)$ explicitly, as $\mathfrak{t}(s,m;\alpha'(s))$. The pertinent observation is that for fixed $m \neq 0$, the function $\alpha'(s) \mapsto \mathfrak{t}(s,m;\alpha'(s))$ is strictly convex. This convexity can be seen in a simple plot of the function $\mathfrak{t}$, as in Fig. 9 (right), but can also be recognized in the geometry of rays that are rotated with respect to each other (Fig. 9 (left)).

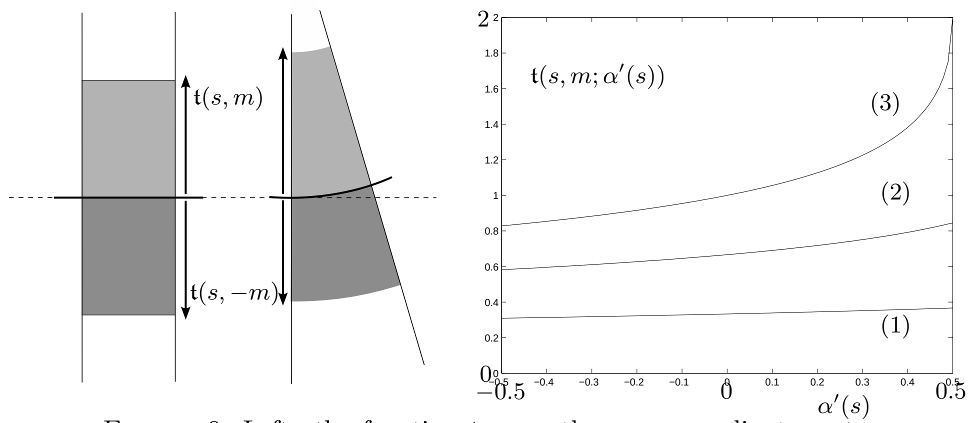

FIGURE 9. Left: the function $\mathfrak{t}$ maps the mass coordinate m to a length coordinate t, in the following way. Fixing an infinitesimal section ds of interface, a section of mass mds, delimited by ds and by two rays, extends away from the interface by a distance of $\mathfrak{t}(s,m;\alpha'(s))$. Right: the function $\alpha'(s) \mapsto \mathfrak{t}(s,m;\alpha'(s))$ (see (7.54)), plotted for $m \in \{1/3, 2/3, 1\}$ (curves (1–3)). Here we have taken $\varepsilon = 1$ and $\sin\beta(s) = 1$.

With this convexity we can understand why curvature is penalized. In (7.75) the relevant expression is

$$\int_0^{M(s)} \mathfrak{t}(s,m;\alpha'(s))\,dm + \int_0^{-M(s)} \mathfrak{t}(s,m;\alpha'(s))\,dm$$

which can be written as

$$\int_0^{M(s)} \big[\mathfrak{t}(s,m;\alpha'(s)) - \mathfrak{t}(s,-m;\alpha'(s))\big]\,dm.$$

This integral describes the transport cost for the transport along an individual ray at position s. Using a symmetry property of $\mathfrak{t}$ and the strict convexity mentioned above,

$$\begin{aligned}
\mathfrak{t}(s,m;\alpha'(s)) - \mathfrak{t}(s,-m;\alpha'(s)) &= \mathfrak{t}(s,m;\alpha'(s)) + \mathfrak{t}(s,m;-\alpha'(s)) \\
&> 2\mathfrak{t}(s,m;0) \\
&= \mathfrak{t}(s,m;0) - \mathfrak{t}(s,-m;0).
\end{aligned}$$

It is in this inequality that we see how curving the interface (or, more precisely, rotating the rays) creates a higher transport cost and therefore a larger value of the energy.

9.3. **Connections to other work: continuum models and phase separation.** Beyond the direct context of lipid bilayers, this work is at the intersection of a variety of different lines of research. There is of course a long-running tradition of slender-body limits in solid mechanics, resulting in a wide variety of simplified models (see e.g. [30] for an overview of the case of elastic plates and shells). Although our final result is rather similar—the Gamma-limiting energy $\mathcal{W}$ is the classical elastica bending energy—the current work is fundamentally different, in that the material at the $\varepsilon > 0$ level is closer to a fluid than a solid: particles may be arbitrarily rearranged without incurring an energy penalty. Indeed, the penalization of curvature that we find in the limit is an effect of *global* geometry rather than of (local) deformation. One might compare this with the difference between an elastic ball and a drop of water; both resist deformation, but for the ball this arises from local intermolecular forces, while the form of the drop results from the global effect of surface area minimization.

Turning to fluid-like systems, the simple fact that the mass represented by u and v concentrates onto a curve is in itself remarkable. Phase separation naturally leads to penalization of interfacial length, as reflected by the term $\int |\nabla u|$ in $\mathcal{F}_\varepsilon$, and the distance penalization by the term $d_1(u, v)$ is necessary to prevent bulk-scale separation. To our knowledge the current work is the first example of a phase-separating system that concentrates on low-dimensional sets.

Concentration onto low-dimensional sets is a commonly observed feature in many (often non-conservative, non-variational) reaction-diffusion systems. For domains with sufficient symmetry a soliton- or pulse-like solution in one dimension can be interpreted as a solution in higher dimensions that concentrates on a hyperplane. In recent years less trivial examples of low-dimensional concentration have been uncovered [20, 9, 2, 3, 4, 23, 37, 38, 36]. Those concentrated solutions are often unstable, however, and sometimes even highly unstable (e.g.[37, 38, 36]), thus strongly contrasting with the stable nature of the structures of this paper.

9.4. **Connections to other work: The elastica functional.** Originally the elastica functional was introduced by Daniel Bernoulli and Euler as the bending energy of an elastic rod, but it has many other applications in different fields. In variational methods for image reconstruction it is widely used [39]. In the theory of phase separations the elastica functional appears as the two dimensional reduction of the Willmore functional (1.4).

It is often natural to consider the elastica functional as acting on boundaries of subsets of the ambient space. In [11] the lower semicontinuity of the elastica functional under L^1-convergence of sets is investigated; see also [44] for a generalization to arbitrary dimensions. The Gamma-convergence of a diffuse interface approximation of the elastica, and more generally the Willmore functional, was conjectured by De Giorgi [22] and proved in a modified form in [43] for space dimensions two and three.

Our approach is not restricted to curves which are boundaries of sets, and therefore open curves can be represented in this framework. A related approach can be found in [15], where functionals which act on phase-field approximations of 'thickened curves' are considered and the Gamma-convergence of these functionals is proved in a topology based on the Haussdorf metric. The elastica functional is part of the limit, however the approximation is completely different.

9.5. **Relevance to the understanding of lipid bilayers.** The derivation that leads from a self-consistent mean-field theory for two-bead copolymers to the energy $\mathcal{F}_\varepsilon$ (Appendix A) contains a number of highly suspicious assumptions. The most glaring one is the severing of the head-tail bond in individual polymers: under this assumption a given head does not keep track of to *which* tail it is connected, as long as it is connected to exactly one tail. The Monge-Kantorovich distance $d_1(u, v)$ is the mathematical implementation of this assumption, and this term is all that keeps heads and tails from separating to arbitrarily large distances.

The most interesting conclusion, from the point of view of applications, might be that this very weak remnant of the covalent bond suffices; that apparently *very little* is needed to convert a system exhibiting bulk-scale phase separation (e.g. an oil-water mixture) into a system which separates at a smaller scale.

9.6. **Other approaches to the upscaling limit.** In the physico-chemical literature many authors have established connections between the Helfrich Hamiltonian (1.3) on one hand and various smaller-scale models of lipid bilayers on the other [47, 41, 34]. Besides being an interesting scientific issue, such a connection should also give insight in the dependence of the coefficients k and $\overline{k}$ on molecular parameters; indeed the derivation given by Helfrich and others is phenomenological, based on scaling and dimensionality arguments, and therefore the coefficients are unknown.

An approach that comes close to ours is that of [18], in which the authors connect the Helfrich description with a model in which lipids are represented by a density and a vectorial quantity that may be interpreted as an alignment order parameter. As in the examples above, however, the final coefficients are again found by a numerical method (minimization of a grand potential energy). Our result here appears to be unique in giving a fully analytic formula for the bending modulus, which is not so surprising since the model contains no unscalable parameters other than ε.

Another way in which our contribution differs from existing work is the possibility to assess the energy penalties associated with non-optimal thickness and with curve 'ends'. In the current paper we only show that these penalities are larger than $O(\varepsilon^2)$, but determining the actual scale of penalization is an interesting open question.

9.7. **Generalization to $d_p(u, v)$ for $p > 1$.** The derivation of the energies $\mathcal{F}_\varepsilon$ presented in Appendix A suggests substituting the Monge-Kantorovich distance by the 2-Wasserstein distance. Some of the results of this paper carry over to this situation, and in fact to the d_p-metric for all $1 < p < \infty$. This follows from the remark that

$$\begin{aligned} d_1(u, v) &= \min \Big(\int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y| \, d\gamma(x, y) \Big) \\ &\le \min \Big(\int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^p \, d\gamma(x, y) \Big)^{1/p} \Big(\int_{\mathbb{R}^2 \times \mathbb{R}^2} d\gamma(x, y) \Big)^{(p-1)/p} \\ &= d_p(u, v), \end{aligned}$$

where we take total mass M equal to one for simplicity. Those statements of Theorem 4.1 that only depend on the upper bound (4.4) therefore carry over without

change for functionals such as

$$\mathcal{F}_\varepsilon^p(u,v) = \varepsilon \int |\nabla u| + \frac{1}{\varepsilon} d_p(u,v)$$
$$\tilde{\mathcal{F}}_\varepsilon^p(u,v) = \varepsilon \int |\nabla u| + \frac{1}{p\varepsilon^p} d_p(u,v)^p$$

To be concrete, sequences $(u_\varepsilon, v_\varepsilon)$ along which the associated functionals $\mathcal{G}_\varepsilon^p$ or $\tilde{\mathcal{G}}_\varepsilon^p$ remain bounded will converge along subsequences to a system of closed $W^{2,2}$-curves.

The Gamma-convergence result of this paper, however, can not be proved in this way: there is a gap between this lower bound and the upper bound that follows from a generalization of Section 8; this gap results from the strict convexity of the function $z \mapsto z^p$ for $p > 1$. Nonetheless, we believe a similar result to be true for all $1 < p < \infty$.

9.8. **Generalizations.** There are several other interesting directions in which further research could continue, among them

- generalizations to higher dimension and higher codimension,
- characterisation of local minimizers of the energies $\mathcal{F}_\varepsilon$,
- formulation of consistent evolution equations on the mesoscale, *i.e.* for the densities $(u_\varepsilon, v_\varepsilon)$.

In higher dimensions, and for limit structures of dimension larger than one, a concept similar to the 'systems of curves' will probably not be appropriate, and we expect that a varifold approach will be better suited. Some results of the present paper are easily generalized but there are also fundamental differences.

To investigate how close the proposed mesoscale approximation is to the Helfrich energy or to the elastica functional it is interesting to compare (local) minimizers of the energy $\mathcal{F}_\varepsilon$ to the variety of shapes found as local minimizers for the Helfrich energy.

A natural idea for extending the mesoscale description of lipid bilayers to a time-dependent evolution is to set up a gradient flow for the energies $\mathcal{F}_\varepsilon$, such as in [28] or [14]. The latter corresponds to a gradient flow based on the 2-Wasserstein distance.

Appendix A. Derivation of $\mathcal{F}_\varepsilon$

The functional $\mathcal{F}_\varepsilon$ that is the basis of this paper arises in a simple model for a water-lipid system. In this model a lipid is represented by a two-bead chain: a head bead and a tail bead connected by a spring. The water molecules are represented by a third type of bead.

The state space at the microscopic level is given by the positions X_t^i, X_h^i, and X_w^j of the lipid tail, lipid head, and water beads; the lipids are numbered by $i = 1, \dots, N_\ell$ and the water beads by $j = 1, \dots, N_w$. Assuming that the beads are confined to a space $\Omega \subset \mathbb{R}^d$, the full microscopic state space for the system is then

$$\mathcal{X} = \Omega^{2N_\ell + N_w}.$$

Elements $X = (X_t^1, \dots, X_t^{N_\ell}, X_h^1, \dots, X_h^{N_\ell}, X_w^1, \dots, X_w^{N_w}) \in \mathcal{X}$ are called microstates.

We describe the system in terms of probabilities on $\mathcal{X}$, i.e. the (probabilistic) state ψ is a probability measure on $\mathcal{X}$:

$$\psi \in \mathcal{E}, \qquad \text{where} \qquad \mathcal{E} = \left\{ \psi : \mathcal{X} \to \mathbb{R}^+, \quad \int_{\mathcal{X}} \psi = 1 \right\}.$$

We assume that neither the microstates themselves nor the measure ψ can be observed at the continuum level; the observables are three derived quantities, the volume fractions of tails, of heads, and of water.

For a given probability measure $\psi \in \mathcal{E}$, the water volume fraction $r_w(\psi) : \Omega \to \mathbb{R}^+$ is defined by

$$r_w(\psi)(x) = v \sum_{j=1}^{N_w} \int_{\mathcal{X}} \psi \, \delta(X_w^j - x) \, dX \qquad \text{for all } x \in \Omega.$$

Here v is the volume fraction of a single bead. The tail and head volume fractions $r_t(\psi)$ and $r_t(\psi)$ are defined similarly.

To specify the behaviour of the system we introduce two free energy functionals. The 'ideal' free energy $F^{\mathrm{id}} : \mathcal{E} \to \mathbb{R}$ models the effects of entropy and the interactions between beads of the same molecule; the 'non-ideal' free energy $F^{\mathrm{nid}} : \mathcal{E} \to \mathbb{R}$ represents the interactions between the beads of different molecules. The total free energy is the sum of the two,

$$F(\psi) = F^{\mathrm{id}}(\psi) + F^{\mathrm{nid}}(\psi).$$

For F^{id} we assume zero temperature and postulate

$$F^{\mathrm{id}}(\psi) = \int_{\mathcal{X}} \psi H^{\mathrm{id}},$$

where the function $H^{\mathrm{id}} : \mathcal{X} \to \mathbb{R}$ is the internal energy of a microstate, and the superscript 'id' again refers to a restriction to interaction within a single lipid molecule. While remarking that many different choices are found in the literature, here we choose simply to implement a spring by penalizing head-tail distance,

$$H^{\mathrm{id}}(X) = \frac{k}{2} \sum_{i=1}^{N_\ell} \left| X_t^i - X_h^i \right|^p.$$

A natural choice is $p = 2$, which gives the energy of a linear spring. Even more realistic seems a general distance term that becomes infinite if $\left|X_t^i - X_h^i\right|$ exceeds a certain maximal value.

For the non-ideal free energy F^{nid} we make an important simplifying assumption: F^{nid} can be written as a function of only the observables,

$$F^{\mathrm{nid}}(\psi) = F^{\mathrm{nid}}(r_w(\psi), r_h(\psi), r_t(\psi)).$$

Typical terms in the non-ideal energy F^{nid} are a convolution integral, in which proximity of hydrophilic (heads and water) beads and hydrophobic tail beads is penalized:

$$\mu \int_\Omega \int_\Omega \big(r_w(\psi)(x) + r_h(\psi)(x)\big)\, r_t(\psi)(y)\, \kappa(x-y)\, dxdy,$$

and a compressibility term that penalizes deviation from unit total volume:

$$\frac{K}{2} \int_\Omega \big(r_w(\psi)(x) + r_t(\psi)(x) + r_h(\psi)(x) - 1\big)^2\, dx.$$

In these expressions we take two limits:

- In the limit $K \to \infty$, we find that the observables satisfy an incompressibility condition
$$r_t(\psi) + r_h(\psi) + r_w(\psi) \equiv 1. \tag{A.1}$$
With this condition the convolution integral above can be written as
$$\mu \int_\Omega \int_\Omega \big(1 - r_t(\psi)(x)\big)\, r_t(\psi)(y)\, \kappa(x-y)\, dxdy. \tag{A.2}$$
- Replacing the fixed function κ by a rescaled version κ_δ, defined by
$$\kappa_\delta(x) := \delta^{-d}\kappa(x/\delta),$$
the integral (A.2) Gamma-converges in the following way,
$$\frac{2}{\delta} \int_\Omega \int_\Omega \big(1 - r_t(\psi)(x)\big)\, r_t(\psi)(y)\, \kappa_\delta(x-y)\, dxdy \xrightarrow{\delta\to 0} K_{1,d} \left(\int \kappa\right) F^{\mathrm{int}}\big(r_t(\psi)\big)$$
where the limit interface functional F^{int} is given by
$$F^{\mathrm{int}}(u) = \begin{cases} \int_\Omega |\nabla u|\, dx & \text{if } u \in \mathrm{BV}\big(\Omega; \{0,1\}\big) \\ \infty & \text{otherwise.} \end{cases} \tag{A.3}$$
Here $K_{1,1} = 1$, $K_{1,2} = 2/\pi$, and $K_{1,3} = 1/2$ [1, 21].

Putting the ideal and non-ideal energies together, and applying the simplifications above, we find (up to rescaling)

$$F(\psi) := F^{\mathrm{id}}(\psi) + F^{\mathrm{nid}}(\psi) = \int_{\mathcal{X}} \sum_{i=1}^{N_\ell} \left|X_t^i - X_h^i\right|^p \psi(X)\, dX + F^{\mathrm{int}}\big(r_t(\psi)\big).$$

We now assume that we minimize F under all variations of ψ that conserve the observables $r_{t,h,w}(\psi)$. By a convexity argument, using the fact that the functions $r_{t,h,w}$ do not distinguish between different lipids or water molecules, it follows that all lipids are identically distributed, as are all water molecules:

$$\psi(X) = \prod_{i=1}^{N_\ell} \psi_\ell(X_t^i, X_h^i) \prod_{j=1}^{N_w} \psi_w(X_w^j),$$

and that the total energy becomes a function of the observables alone, by minimization over all other degrees of freedom (again we disregard rescaling)

$$F(\rho_t, \rho_h, \rho_w) := F^{\rm int}(\rho_t) + \\ + \inf_{\psi_\ell} \Big\{ \int_{\Omega\times\Omega} |X_t - X_h|^p\, \psi_\ell(X_t, X_h)\, dX_t dX_h : r_t(\psi_\ell) = \rho_t,\ r_h(\psi_\ell) = \rho_h \Big\}$$

with the remaining constraint $\rho_t + \rho_h + \rho_w \equiv 1$. Remark that the final infimum is exactly the Wasserstein distance of degree p, $d_p(\rho_t, \rho_h)^p$.

We now recover the functional $\mathcal{F}_\varepsilon$ in (1.1) for $\varepsilon = 1$ by defining $u := \rho_t$, $v := \rho_h$ and by choosing $p = 1$. Combining the pure-phase condition $u(x) = \rho_t(x) \in \{0, 1\}$ for all x (A.3) with the incompressbility condition (A.1) gives $v(x) = \rho_h(x) \in [0, 1]$ and $uv = 0$. It is easy to see that there is nothing to be gained in letting v take values in the interior $(0, 1)$, and we can therefore consider $\mathcal{F}_1$ to be given on the set of admissible functions

$$\mathcal{K}_1 := \Big\{ (u, v) \in \mathrm{BV}\big(\mathbb{R}^2; \{0, 1\}\big) \times L^1\big(\mathbb{R}^2; \{0, 1\}\big) : \\ \int u = \int v = M,\ uv = 0 \text{ a.e.} \Big\}.$$

The remaining dependence on ε is a consequence of simple rescaling.

Appendix B. Ring solutions

We consider $r_1 < r_2 < R < r_3 < r_4$ with

$$r_2^2 = \tfrac{1}{2}(R^2 + r_1^2), \qquad r_3^2 = \tfrac{1}{2}(R^2 + r_4^2), \tag{B.1}$$

$$2R = r_1 + r_4 \tag{B.2}$$

and u, v given by

$$u = \mathcal{X}_{B_{r_3}(0)\setminus B_{r_2}(0)}, \qquad v = \mathcal{X}_{B_{r_2}(0)\setminus B_{r_1}(0)} + \mathcal{X}_{B_{r_4}(0)\setminus B_{r_3}}.$$

In this setting the unique monotone optimal transport map S from u to v is radially symmetric, $S(x) = S(|x|)$ and determined by the requirement that

$$\int_{r_2}^{s} 2\pi r\, dr = \int_{r_1}^{S(s)} 2\pi r\, dr \qquad \text{for } r_2 < s < R, \tag{B.3}$$

$$\int_{s}^{r_3} 2\pi r\, dr = \int_{S(s)}^{r_4} 2\pi r\, dr \qquad \text{for } R < s < r_3. \tag{B.4}$$

Then

$$\begin{aligned} d_1(u, v) &= \int |x - S(x)| u(x)\, dx \\ &= \int_{r_2}^{R} 2\pi r(r - S(r))\, dr + \int_{R}^{r_3} 2\pi r(S(r) - r)\, dr. \end{aligned} \tag{B.5}$$

To compute the first integral we introduce new coordinates

$$m(r) = \pi r^2 - \pi r_2^2 \quad \text{with inverse} \quad r(m) = \big(r_2^2 + \frac{1}{\pi} m\big)^{1/2}. \tag{B.6}$$

In this coordinates (B.3) implies

$$m(r) \;=\; m(S(r)) - m(r_1), \tag{B.7}$$

$$S(r(m)) \;=\; r\big(m + m(r_1)\big) \;=\; r\big(m - m(R)\big), \tag{B.8}$$

where in the last equality we have used that

$$m(r_1) + m(R) \;=\; \pi\big(r_1^2 - 2r_2^2 + R^2\big) \;=\; 0$$

by (B.1). The first integral in (B.5) therefore becomes

$$\begin{aligned}
\int_{r_2}^{R} 2\pi r(r - S(r))\,dr \;&=\; \int_0^{m(R)} \big(r(m) - r(m - m(R))\big)\,dm \\
&= \int_0^{m(R)} r(m) - r(-m)\,dm \\
&= \frac{2\pi}{3}\Big[\Big(r_2^2 + \frac{1}{\pi}m\Big)^{3/2} + \Big(r_2^2 - \frac{1}{\pi}m\Big)^{3/2}\Big]_0^{m(R)} \\
&= \frac{2\pi}{3}\Big[\Big(r_2^2 + \frac{1}{\pi}m(R)\Big)^{3/2} + \Big(r_2^2 - \frac{1}{\pi}m(R)\Big)^{3/2} - 2r_2^3\Big].
\end{aligned} \tag{B.9}$$

If we now introduce $t = (r_4 - r_1)/2$ then

$$R - r_1 \;=\; r_4 - R \;=\; t, \qquad m(R) \;=\; \frac{\pi}{2}(2Rt - t^2),$$

and (B.9) yields

$$\int_{r_2}^{R} 2\pi r(r - S(r))\,dr \;=\; \frac{2\pi}{3}\Big[R^3 + (R-t)^3 - 2R^3\Big(1 - \frac{t}{R} + \frac{t^2}{2R^2}\Big)^{\frac{3}{2}}\Big]. \tag{B.10}$$

If we do the analogous calculations for the second integral in (B.5) we obtain that

$$\int_{R}^{r_3} 2\pi r(S(r) - r)\,dr \;=\; \frac{2\pi}{3}\Big[R^3 + (R+t)^3 - 2R^3\Big(1 + \frac{t}{R} + \frac{t^2}{2R^2}\Big)^{\frac{3}{2}}\Big]. \tag{B.11}$$

which is the expression on the right-hand side of (B.10) with t substituted by $-t$. We consider $R \to \infty$ with t of order one and obtain by a Taylor expansion in $1/R$

$$\begin{aligned}
&d_1(u,v) + \int |\nabla u| \\
&= \frac{2\pi}{3}\Big[2R^3 + (R+t)^3 + (R-t)^3 - 2R^3\Big(1 + \frac{t}{R} + \frac{t^2}{2R^2}\Big)^{\frac{3}{2}} - 2R^3\Big(1 - \frac{t}{R} + \frac{t^2}{2R^2}\Big)^{\frac{3}{2}}\Big] \\
&\quad + 2\pi R\Big[\Big(1 + \frac{t}{R} + \frac{t^2}{2R^2}\Big)^{\frac{1}{2}} + \Big(1 - \frac{t}{R} + \frac{t^2}{2R^2}\Big)^{\frac{1}{2}}\Big] \\
&= \pi R t^2 + \frac{\pi}{16}\frac{t^4}{R} + 4\pi R + \frac{\pi}{2}\frac{t^2}{R} + O(R^{-3}) \\
&= 2\pi R t\Big[\Big(\frac{t}{2} + \frac{2}{t}\Big) + \frac{1}{4R^2} + \frac{(2-t)}{32R^2}\big(t^2 + 2t - 4\big) + O(R^{-2})O(|t-2|)\Big] \\
&= 2\pi R t\Big[2 + \frac{1}{4}(t-2)^2 + \frac{1}{4}R^{-2} + O(R^{-2})O(|t-2|)\Big],
\end{aligned}$$

which is the desired asymptotic development.

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